Math 318 - Geometry/Topology 2

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Introduction

Math 318 is one of the nine courses offered for first-year mathematics graduate students at the University of Chicago. It is the second of three courses in the year-long geometry/topology sequence.

These notes are being live-TeXed, though I edit for typos and add diagrams requiring the Ti*k*Z package separately. I am using the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to `chonoles@math.uchicago.edu`.

Lecture 1 (2013-01-07)

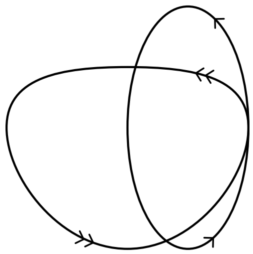
The last time I taught this course was in 2002. Back then, differential topology was taught first. Here is what I covered back then, together with some book recommendations:

- For basics (e.g., the definition of differentiable manifold), I recommend Chapter 1 of Frank Warner's *Foundations of Differentiable Manifolds and Lie Groups.*
- I also covered the notion of "general position". A good reference for this is Chapter 1 of Milnor's *Topology from the Differentiable Viewpoint.* The book by Guillemin and Pollack is also good for this.
- Hirsch's *Differential Topology* is good for everything.
- The final main topic I covered last time was de Rham cohomology, the Poincaré lemma, etc.

General Position

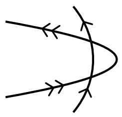
Given $f_0 : M \to N$, how beautiful can a perturbation f (of f_0) get?

For example, consider the map $f_0 : \mathbb{S}^1 \to \mathbb{R}^2$ with the following graph:



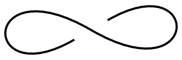
The focus of our attention is at the point of self-tangency. We have two options: we can make the curve avoid itself,

or we can make it intersect itself in a non-tangent way,



We might also consider a map $f_0 : \mathbb{S}^1 \to \mathbb{R}^3$ that intersects itself:

But because we are working in $\mathbb{R}^3$, there is no reason it should intersect itself at all; we can just perturb the curve at the intersection appropriately.



This is an illustration of the Whitney theorem; given an n-dimensional manifold, we can always find an embedding in $\mathbb{R}^k$ for $k \geq 2n+1$, and an immersion for $k = 2n$. This result requires Sard's theorem. These theorems fall under the heading of general position.

Here are some other results I plan to talk about:

- Given a map $f_0 : M \to N$ and a closed submanifold $A \hookrightarrow N$, the preimage $f_0^{-1}(A)$ may not be a submanifold of M, but after an appropriate perturbation, $f^{-1}(A)$ is a submanifold.
- A closed submanifold $A \hookrightarrow N$ of codimension r defines a cohomology class $\theta(A) \in H^r(N; \mathbb{Z}/(2))$. If its normal bundle is oriented, we actually get a cohomology class $\theta(A) \in H^r(N; \mathbb{Z})$.
- Let $f : M \to N$ be a map. If f is transverse to A, then $f^{-1}(A)$ is also a submanifold and $\theta(f^{-1}(A)) \in H^r(M; \mathbb{Z}/(2))$ and $f^*(\theta(A)) = \theta(f^{-1}(A))$. If M and A are both submanifolds of N that meet transversally, then $M \cap A$ is a submanifold of N and $\theta(A) \smile \theta(M) = \theta(A \cap M)$.

This whole course is repeated applications of three statements, two of which you've seen already:

- Inverse Function Theorem
- Existence and Uniqueness of Solutions to ODEs
- Sard's Theorem

Differentiable Manifolds

As you should all know, a topological manifold is a space which is locally homeomorphic to $\mathbb{R}^n$.

We can't consistently decide when a function on a manifold is differentiable. To fix this problem, we give a manifold a differentiable structure. The usual definition of a differentiable structure on a topological manifold is via coordinate charts. However, if you work with complex algebraic geometry, a more natural definition is with sheaves. You can find the definition of sheaf, subsheaf, etc. in Warner's book, or in Gunning's *Lectures on Riemann Surfaces.*

Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $f : \Omega \to \mathbb{R}$ is k-times continuously differentiable, and we say that f is C^k, when the following holds:

(a) If $k = 0$, f is continuous, and

(b) if $k > 0$, then $\partial_i f(x)$ are defined for all $1 \leq i \leq n$ and all $x \in \Omega$, and $\partial_i f : \Omega \to \mathbb{R}$ is C^{k-1}.

We say that f is C^∞ when f is C^k for all k. We can define the notions of C^k, C^∞, and C^ω manifolds (the notation C^ω means real analytic, i.e., the Taylor series equals the function). For any topological space X, we will write $C^0(X)$ for the set of real-valued continuous functions on X, and C^0_X for the sheaf of such functions on X.

Given an open $\Omega \subset \mathbb{R}^n$, we can consider the set $R = C^k(\Omega)$ of C^k functions on Ω (or $C^\infty(\Omega)$ or $C^\omega(\Omega)$, respectively). Observe that

(1) R is a ring under pointwise addition and multiplication,

(2) all constant functions are in R, and

(3) R is a subsheaf of the sheaf of continuous functions C^0_Ω; that is, for all $\Omega' \subseteq \Omega$ we have $R(\Omega') \subseteq C^0(\Omega')$, and the restriction of an element of $R(\Omega)$ to Ω' is an element of $R(\Omega')$.

Thus, in fact, R is a sheaf of $\mathbb{R}$-algebras.

Given a pair (M, R) where M is a topological space and R is a subsheaf of C^0_M, we say that a pair (U, ψ) is a coordinate chart when $U \subset M$ is open and $\psi : U \to \mathbb{R}^m$ is a homeomorphism onto its image $\psi(U)$, which we require to be open in $\mathbb{R}^m$.

We say that a coordinate chart (U, ψ) is admissible if, for all open $U' \subset U$ and $h \in C^0(U')$,

$$h \in R(U') \iff h \circ (\psi|_{U'})^{-1} \in C^\infty(\psi(U')).$$

(Note that the map $\psi|_{U'} : U' \to \psi(U')$ is a homeomorphism because ψ was a homeomorphism of U onto its image.) Lastly, we say that M is a C^∞ manifold if it is covered by admissible coordinate charts. The definitions of C^k and C^ω manifolds are analogous.

▼ What is the redundancy, exactly? ▼

This definition actually has a redundancy in the case of C^∞ manifolds. This is because for C^∞ manifolds, we have partitions of unity available to us.

But I'm using the more general definition because other objects, such as real analytic or complex analytic manifolds, do not have partitions of unity.

Let's review:

1. We've defined the notion of a coordinate chart for any topological space M.
2. Given a pair (M, R) where $R \subseteq C^0_M$, we defined what it means for a chart to be admissible.
3. (M, R) is a C^∞ manifold if M is covered by admissible charts.

When discussing a given C^∞ manifold M, we will refer to its sheaf as C^∞_M.

Lecture 2 (2013-01-09)

Review of Previous Class

For any topological space Y, we define $C^0(Y)$ to be the set of continuous real-valued functions on Y. For any open subsets $U \subseteq V$ of Y, there is a restriction map $\mathrm{Res}(U, V) : C^0(V) \to C^0(U)$. Recall that a presheaf R of subsets of C^0_Y consists of, for all open $U \subseteq Y$, a choice of $R(U) \subseteq C^0(U)$ such that whenever $U \subseteq V$, we have $\mathrm{Res}(U, V)(f) \in R(U)$ for any $f \in R(V)$.

Given a presheaf R of subsets of C^0_Y, we say that it is a sheaf of subsets when the following condition is satisfied: given any open $V \subseteq Y$ and $f \in C^0(V)$, if for all $p \in V$ there is a neighborhood $V_p \subseteq V$ of p such that $\mathrm{Res}(V_p, V)(f) \in R(V_p)$, then $f \in R(V)$.

We say that a sheaf R of subsets of C^0_Y is a sheaf of $\mathbb{R}$-subalgebras if $R(U) \subseteq C^0(U)$ is an $\mathbb{R}$-subalgebra for all open U.

For us, a ringed space will consist of a pair (Y, R) of a topological space Y and a sheaf R of $\mathbb{R}$-subalgebras of C^0_Y. A morphism $(Y, R) \to (Y', R')$ of ringed spaces is a continuous map $f : Y \to Y'$ such that for all open $V' \subseteq Y'$ and $g \in R'(V')$, the pullback f^*g, i.e. the composition

$$f^{-1}(V') \xrightarrow{f|_{f^{-1}(V')}} V' \xrightarrow{g} \mathbb{R},$$

is an element of $R(f^{-1}(V))$. An isomorphism $(Y, R) \to (Y', R')$ is a homeomorphism $f : Y \to Y'$ such that $f^{-1} : (Y', R') \to (Y, R)$ is also a morphism of ringed spaces.

Fix some $k \in \{0, 1, 2, \ldots, \infty, \omega\}$. A standard example of a ringed space is (Ω, C^k_Ω) where Ω is an open subset of $\mathbb{R}^n$, and C^k_Ω is the sheaf defined by $C^k_\Omega(V) = C^k(V)$ for any open $V \subseteq \Omega$.

Definition. A C^k manifold is a ringed space (M, R) such that M is covered by open sets U_i where each (U_i, R_{U_i}) is isomorphic as a ringed space to some $(\Omega_i, C^k_{\Omega_i})$, where Ω_i is an open subset of $\mathbb{R}^{n_i}$. We then refer to R as C^k_M.

Given C^k manifolds M and N, a map $f : M \to N$ is a C^k map when it is a morphism of ringed spaces $(M, C^k_M) \to (N, C^k_N)$.

Tangent Vectors

A natural way we think about tangent vectors arising is as the tangent vector to a curve. We can define an equivalence relation $\sim$ on curves through a point $p \in M$, according to when they have the same tangent vector at p.

Given a C^k manifold M and a point $p \in M$, we consider two C^k maps $\gamma : (-a, a) \to M$ and $\delta : (-b, b) \to M$ where $\gamma(0) = \delta(0) = p$. When do we want to say that $\gamma \sim \delta$?

First, let's define germs. If $U, V \subseteq M$ are open subsets containing p and $f \in C^k_M(U)$ and $g \in C^k_M(V)$, we say that $f \sim g$ if there is an open $W \subseteq U \cap V$ containing p such that $f|_W = g|_W$. The collection

$$C^k_{M,p} := \coprod_{\text{open } U \ni p} C^k_M(U) \Big/ \sim$$

is the set of germs of functions on M at p. In fact, it is an $\mathbb{R}$-algebra.

Given a germ f at p, say defined on U (so that $f : U \to \mathbb{R}$ is a C^k function), there is some $0 < a' < a$ such that the image of $\gamma|_{(-a',a')}$ is contained in U, so we can consider the composition

$$(-a', a') \xrightarrow{\gamma|_{(-a',a')}} U \xrightarrow{f} \mathbb{R}.$$

The composition is defined on a (smaller) neighborhood of 0, so we can still consider $(f \circ \gamma)'(0)$.

The above procedure gives a linear functional $D_\gamma : C^k_{M,p} \to \mathbb{R}$, assigning to a germ f the value of $(f \circ \gamma)'(0)$. We say that $\gamma \sim \delta$ precisely when $D_\gamma = D_\delta$. A tangent vector of M at p is then an equivalence class of curves under this relation.

Assume that M is an open set $\Omega \subseteq \mathbb{R}^n$. Then the chain rule tells us that $(f \circ \gamma)'(0) = f'(p)\gamma'(0)$, where $f'(p) : \mathbb{R}^n \to \mathbb{R}$ is a linear functional. Then in this case, $\gamma_1 \sim \gamma_2 \iff \gamma_1'(0) = \gamma_2'(0)$, so this matches our intuition about how tangent vectors should behave. We see that the space of tangent vectors of Ω at p can be canonically identified with $\mathbb{R}^n$.

Note that there is no natural notion of adding curves, but when we think about the corresponding linear functionals D_γ, the vector space structure becomes clear. According to our definition, D_γ is an element of the dual space of $C^k_{M,p}$. Thus, by definition, the tangent space of M at p can be considered as a subset of the dual space, i.e. $T_pM \hookrightarrow (C^k_{M,p})^*$.

Theorem. *T_pM is a linear subspace of $(C^k_{M,p})^*$.*

Proof. It suffices to check in the case that $M =$ an open subset $\Omega \subseteq \mathbb{R}^n$. In this case, we see that

$$T_pM = \{\textstyle\sum_{i=1}^n a_i \partial_i(p)\},$$

where for any germ $f \in C^k_{M,p}$, the functional $\partial_i|_p$ sends $f \mapsto \partial_i f|_p := \frac{\partial f}{\partial x_i}|_p$. ☐

Note that for any curve γ, the corresponding functional $D = D_\gamma$ satisfies the Leibniz condition:

$$D(fg) = f(p)Dg + g(p)Df.$$

Thus, it is reasonable to include this condition in our definition of tangent vector.

Proposition. *For any $p \in \mathbb{R}^n$, the only linear functionals on $C^\infty_{\mathbb{R}^n,p}$ that satisfy the Leibniz condition are those of the form $f \mapsto \sum_{i=1}^n a_i \partial_i f|_p$.*

▼ What is a counterexample? ▼

You might find it interesting to show that the proposition is false even for $C^1_{\mathbb{R}}$.

Proof of Proposition. WLOG, let's say that $p = 0$. First, we make two observations:

1. If we put $f = 1$ in the Leibniz condition, we get that $D(g) = D(g) + g(p)Df$ for all g, so we must have $D(1) = 0$, and hence by linearity, $D(\lambda) = 0$ for any $\lambda \in \mathbb{R}$.
2. $\partial_i(x_j) = \delta_{ij}$.

Let D be a functional that satisfies the Leibniz condition, and put $D(x_i) = a_i$. Let

$$E = D - \sum_{i=1}^n a_i \partial_i.$$

We see that $E(x_i) = 0$ for all i, and we want to show that this forces $E = 0$, i.e. $E(f) = 0$ for all $f \in C^\infty_{\Omega,p}$. Equivalently, we can show that $E(f - f(p)) = 0$, since $E(f(p)) = 0$.

Lemma. *If f is a C^∞ function defined in a neighborhood of 0 in $\mathbb{R}^n$ and $f(0) = 0$, then there exist C^∞ functions $g_1, \dots, g_n$ (defined in a possibly smaller neighborhood of 0) such that $f = \sum_{i=1}^n x_i g_i$.*

Intuitively, this lemma makes sense, because if f vanishes along the hyperplane $x_1 = 0$, this suggests that x_1 "divides" f, i.e. that $f(x_1, x_2) = x_1 g(x_1, x_2)$ for some function g; and if f vanishes on the space where $x_1 = x_2 = 0$, this suggests that f is a sum of a multiple of x_1 and a multiple of x_2. Proving this lemma will be a part of the first homework assignment.

Using the lemma, we can conclude the argument. Because E satisfies the Leibniz condition, we have

$$E(f) = \sum_{i=1}^n \underbrace{x_i(0)}_{=0} E(g_i) + g_i(0) \underbrace{E(x_i)}_{=0} = 0. \qquad \square$$

We define the contangent space $T_p^* M$ of M at p to be the dual space $(T_p M)^*$. If $v_1, \dots, v_n$ are a basis for $T_p M \subset (C^k_{M,p})^*$, then letting $D = \bigcap_{i=1}^n \ker(v_i)$, we have $T_p^* M = C^k_{M,p}/D$.

Our proposition implies that when $k = \infty$, we have $D = \mathbb{R} + \mathfrak{m}^2$ where $\mathfrak{m} = \{f \in C^\infty_{M,p} \mid f(p) = 0\}$.

Lecture 3 (2013-01-11)

From now on, assume all spaces in this course are Hausdorff, unless I say so explicitly. Additionally, we will only be considering the case that $k = \infty$.

You will often see the following definition of a C^∞ manifold:

> **Data:** A topological space M, covered by open sets U_α indexed by $\alpha \in A$, together with homeomorphisms $\psi_\alpha : U_\alpha \to \Omega_\alpha \subset \mathbb{R}^n$, where Ω_α is open, which satisfy
>
> **Conditions:** For any $\alpha, \beta \in A$, the composition $(\psi_\alpha|_{U_\alpha \cap U_\beta}) \circ (\psi_\beta|_{U_\alpha \cap U_\beta})^{-1}$ is a C^∞ map between open subsets of $\mathbb{R}^n$:

$$\begin{tikzcd}
\psi_\beta(U_\alpha \cap U_\beta) \arrow[r, "\psi_\beta|_{U_\alpha \cap U_\beta}", "\text{homeo}"'] & U_\alpha \cap U_\beta \arrow[r, "\psi_\alpha|_{U_\alpha \cap U_\beta}", "\text{homeo}"'] & \psi_\alpha(U_\alpha \cap U_\beta)
\end{tikzcd}$$

This information specifies a ringed space.

Let $U \subset M$ be open, and let $f : U \to \mathbb{R}$ be a function. For any $\alpha \in A$, we can consider the composition $f|_{U \cap U_\alpha} \circ (\psi_\alpha|_{U \cap U_\alpha})^{-1}$,

$$\begin{tikzcd}
\psi_\alpha(U \cap U_\alpha) \arrow[r, "(\psi_\alpha|_{U \cap U_\alpha})^{-1}"] & U \cap U_\alpha \arrow[r, "f|_{U \cap U_\alpha}"] & \mathbb{R}.
\end{tikzcd}$$

We then *define* f to be a C^∞ function on U precisely when the map $f|_{U \cap U_\alpha} \circ (\psi_\alpha|_{U \cap U_\alpha})^{-1}$ is C^∞ for all $\alpha \in A$.

Corollary (of the definition). *For any of the n projections $p_i : \mathbb{R}^n \to \mathbb{R}$, the composition*

$$\begin{tikzcd}
U_\alpha \arrow[r, "\psi_\alpha"] & \Omega_\alpha \arrow[r, hook] & \mathbb{R}^n \arrow[r, "p_i"] & \mathbb{R}
\end{tikzcd}$$

is a C^∞ function on U_α.

Now let's go back to tangent spaces for a moment. Recall that

$$T_pM = \{\text{linear functionals } E : C^\infty_{M,p} \to \mathbb{R} \mid E(fg) = f(p)E(g) + g(p)E(f)\}.$$

Given a C^∞ map of pointed manifolds $\psi : (M,p) \to (N,q)$ – in other words, $\psi : M \to N$ is a C^∞ map and $\psi(p) = q$ – then for any open $V \subset N$ with $q \in V$, we get a pullback map

$$\begin{tikzcd}
C^\infty(V) \arrow[r, "\psi^*"] \arrow[d, "\text{restriction}"'] & C^\infty(\psi^{-1}(V)) \arrow[d, "\text{restriction}"] & \\
C^\infty_{N,q} \arrow[r, "\psi^*"] & C^\infty_{M,p} \arrow[r, "E"] & \mathbb{R}
\end{tikzcd}$$

Thus, given a tangent vector $E \in T_pM$, we get a linear map $(E \circ \psi^*) : C^\infty_{N,q} \to \mathbb{R}$ satisfying the Leibniz rule, and thus $(E \circ \psi^*) \in T_qN$. By definition, we say that $E \circ \psi^* = \psi'(p)E$. The map $\psi'(p) : T_pM \to T_qN$ is a linear transformation.

Submanifolds

One basic thing we need to decide is when a subset A of a C^∞ manifold M is a C^∞ submanifold. For example, if $M = \mathbb{R}^3$, then the cone

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2,\ z \geq 0\}$$

should **not** be a C^∞ submanifold, because it isn't smooth at the apex.

We define

$$R(A) = \left\{ f : A \to \mathbb{R} \;\middle|\; \begin{array}{c} \text{for all } p \in A, \text{ there is some} \\ \text{neighborhood } U_p \subset M \text{ of } p \\ \text{and a } C^\infty \text{ map } f_p : U_p \to \mathbb{R} \\ \text{such that } f|_{A \cap U_p} = f_p|_{A \cap U_p} \end{array} \right\}.$$

If $B \subset A$, then the map $f \mapsto f|_B$ defines a map $R(A) \to R(B)$; this is because the requirement is clearly a local condition. If we apply this to those B which are open in A (that is, in the relative topology), this yields a sheaf R of $\mathbb{R}$-subalgebras of C^0_A (check for yourself that these are $\mathbb{R}$-algebras).

Definition. Given a C^∞ manifold M and a subset $A \subseteq M$, we say that A is a C^∞ submanifold if the ringed space (A, R) defined in this way is a C^∞ manifold. More generally, if $p \in A$, then we say that A is a C^∞ submanifold at p when there is a neighborhood U_p of p in A such that (U_p, R_{U_p}) is a C^∞ submanifold.

Examples. The subset

$$\{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\}$$

is, according to our definition, a C^∞ submanifold of $\mathbb{R}^3$. Similarly,

$$\{(x, \sin(\tfrac{1}{x})) \in \mathbb{R}^2 \mid x > 0\}$$

is a C^∞ submanifold of $\mathbb{R}^2$. Usually, C^∞ submanifolds are required to be closed, and we will add that requirement later, but for now, these are both submanifolds.

Here is another example. Let $\Omega_1 \subset \mathbb{R}^m$ and $\Omega_2 \subset \mathbb{R}^{n-m}$ be open sets, and let $y_0 \in \Omega_2$. If we set $M = \Omega_1 \times \Omega_2$, then $A = \Omega_1 \times \{y_0\}$ is a C^∞ submanifold of M. The open sets of A are precisely those subsets of the form $U \times \{y_0\}$ for some open $U \subset \Omega_1$.

If $i : \Omega_1 \to A$ is the obvious map $i(x) = (x, y_0)$, one observes that $f \in R(i(U))$ if and only if $f \circ i$ is C^∞ in the traditional sense. Thus, $i^{-1} : A \to \Omega_1$ is an admissible coordinate chart covering A, demonstrating that A is a C^∞ manifold in our usual sense.

Theorem. *Let M be a C^∞ manifold of dimension n, and let $p \in A \subset M$. Then A is a C^∞ submanifold of dimension m at p if and only if there exists a neighborhood U_p of p in A and a diffeomorphism $\phi : U_p \to \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^m$ and $\Omega_2 \subset \mathbb{R}^{n-m}$ are open and $y_0 \in \Omega_2$, such that $\phi(U_p \cap A) = \Omega_1 \times \{y_0\}$.*

We've just done the $\Longleftarrow$ direction.

Lemma (Parametric form, special case)**.** *If $U \subseteq \mathbb{R}^m$ is open with $0 \in U$, if $f : (U, 0) \to (\mathbb{R}^n, 0)$ is a C^∞ map, and if $f'(0) : \mathbb{R}^m \to \mathbb{R}^n$ is injective, then there is some open $U_1 \subset U$ with $0 \in U_1$, some open $U_2 \subset \mathbb{R}^{n-m}$ with $0 \in U_2$, some open $\Omega \subset \mathbb{R}^n$, and some diffeomorphism $\phi : U_1 \times U_2 \to \Omega$, such that there is a commutative diagram*

$$\begin{tikzcd} U_1 \arrow[r, "i"] \arrow[rrr, bend right, "f|_{U_1}"'] & U_1 \times U_2 \arrow[r, "\phi"] & \Omega \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

where $i(x) = (x, 0)$ for all $x \in U_1$.

Corollary. *In particular, $f(V)$ is a C^∞ submanifold of $\mathbb{R}^n$ for all open $V \subset U_1$.*

Proof of special case of Lemma 1. One may choose a C^∞ map $g : (W, 0) \to (\mathbb{R}^n, 0)$, where $W \subset \mathbb{R}^{n-m}$ is open, such that $g'(0) : \mathbb{R}^{n-m} \to \mathbb{R}^n$ is injective, and $f'(0)\mathbb{R}^m + g'(0)\mathbb{R}^{n-m} = \mathbb{R}^n$.

Let $F : U \times W \to \mathbb{R}^n$ be the map defined by $F(x, y) = f(x) + g(y)$. Then $F'(0, 0) : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ is an isomorphism. The inverse function theorem implies that there exist U_1 and U_2, neighborhoods of $0 \in U$ and $0 \in W$ respectively, and an open $\Omega \subset \mathbb{R}^n$, such that $F|_{U_1 \times U_2} : U_1 \times U_2 \to \Omega$ is a diffeomorphism. ☐

Lemma (Parametric form)**.** *If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open, if $f : (U, p) \to (V, q)$ is C^∞, and if $f'(q) : \mathbb{R}^m \to \mathbb{R}^n$ is injective, then there are neighborhoods U_1 of q in U, U_2 of 0 in $\mathbb{R}^{n-m}$, and Ω of p in V, and a diffeomorphism $\phi : U_1 \times U_2 \to \Omega$ such that $\phi(x, 0) = f(x)$ for all $x \in U_1$.*

Proposition (Implicit form)**.** *If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^{n-m}$ are open, if $f : (U, p) \to (V, q)$ is C^∞, and if $f'(p) : T_p\mathbb{R}^n \to T_q\mathbb{R}^{n-m}$ is surjective, then there are neighborhoods Ω of p in U, Ω_1 of 0 in $\mathbb{R}^m$, and Ω_2 of q in V, and a diffeomorphism $\phi : \Omega \to \Omega_1 \times \Omega_2$ such that $f(x) = p_2(\phi(x))$ for all $x \in \Omega$, where $p_2 : \Omega_1 \times \Omega_2 \to \Omega_2$ is projection onto the second factor.*

Proof of $\Longrightarrow$ in Theorem. Because the question is local, we may assume WLOG that M is an open subset of $\mathbb{R}^n$.

Recall that we are assuming A is a C^∞ submanifold of dimension m at p. Thus, there is an isomorphism of ringed spaces $g : (U_1 \cap A, R_{U_1 \cap A}) \to (V, C^\infty_V)$ where U_1 is a neighborhood of p in M and V is an open subset of $\mathbb{R}^m$. Let $q = g(p)$, and let $f = g^{-1}$. Let $(f_1, \ldots, f_n)$ be the coordinate functions of the composition

$$\begin{tikzcd} V \arrow[r, "f"] & U_1 \cap A \arrow[r, hook] & U_1 \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

and let the coordinate functions on $\mathbb{R}^n$ be denoted $p_i : \mathbb{R}^n \to \mathbb{R}$. Because the p_i are C^∞, the maps $p_i|_{U_1 \cap A}$ belong to $R(U_1 \cap A)$, and because f is a morphism of ringed spaces, these pull back to C^∞ functions on V. Thus, the functions f_i are all C^∞ functions on V.

Let $(g_1, \ldots, g_m)$ be the coordinate functions of g. Then we have $g_i \in R(U_1 \cap A)$ for all i. It follows that there is a neighborhood U_2 of p in U_1 with C^∞ functions $\widetilde{g}_i$ on U_2 such that $\widetilde{g}_i(x) = g_i(x)$ for all $x \in U_2 \cap A$, which we can put together into $\widetilde{g}$, a C^∞ map $\widetilde{g} : U_2 \to \mathbb{R}^m$ such that $\widetilde{g}(x) = g(x)$ for all $x \in U_2 \cap A$.

Thus, we have produced open subsets $V \subseteq \mathbb{R}^m$ and $U_2 \subseteq \mathbb{R}^n$, and C^∞ maps $f : U \to \mathbb{R}^n$ and $\widetilde{g} : U_2 \to \mathbb{R}^m$, such that $\widetilde{g}(f(x)) = x$ for all $x \in f^{-1}(U_2 \cap A)$. The chain rule shows that $f'(q)$ is injective, and applying the lemma now, we are done. ☐

Lecture 4 (2013-01-14)

Last time, we defined what it means for A to be a C^∞ submanifold of M at p. In the future, I'll sometimes shorten this to "A is smooth at p".

Theorem (Implicit Function Theorem)**.** *Let M be a C^∞ manifold and $p \in M$. Let $U \subset M$ be an open neighborhood of p. Let N be another C^∞ manifold, and let $f : U \to N$ be a C^∞ map. If the derivative $f'(p) : T_pM \to T_{f(p)}N$ is surjective, then the fiber $f^{-1}(f(p))$ is smooth at p.*

Definition. Let $f : M \to N$ be a C^∞ map. We say that $q \in N$ is a regular value of f if for all $p \in f^{-1}(q)$, the derivative $f'(p) : T_pM \to T_qN$ is surjective.

Corollary. *If $q \in N$ is a regular value of $f : M \to N$, then $f^{-1}(q)$ is C^∞ submanifold of M. Furthermore, for all $p \in f^{-1}(q)$, the tangent space $T_pf^{-1}(q)$ can be identified with the kernel of $f'(p) : T_pM \to T_qN$.*

Proof of Implicit Function Theorem. Assume that U is open in $\mathbb{R}^n$ and that $p = 0 \in U$, and assume that N is open in $\mathbb{R}^m$. Let $f : U \to N$, and assume that $f'(0)$ is surjective.

Clearly, there exists a C^∞ map $g : U \to \mathbb{R}^{n-m}$ such that if $F = (f, g)$, then the derivative $F'(0) : T_0U \to T_qN \times T_0\mathbb{R}^{n-m}$ is an isomorphism. By the Inverse Function Theorem, there is an open neighorhood $U' \subseteq U$ of 0, an open neighborhood $N' \subseteq N$ of q, and an open neighborhood $V \subseteq \mathbb{R}^{n-m}$ of 0, such that there is a commutative diagram

$$\begin{tikzcd}
U' \arrow[rr, "F|_{U'}", "\cong"'] \arrow[dr, "f|_{U'}"'] & & N' \times V' \arrow[dl, "p_1"] \\
& N' &
\end{tikzcd}$$

Note that $F|_{U'}$ restricts to a diffeomorphism from $f^{-1}(q) \cap U'$ to $\{q\} \times V'$, and that the latter is a C^∞ submanifold of $N' \times V'$. This proves the theorem.

Now we prove the corollary. The commutative diagram

$$\begin{tikzcd}
f^{-1}(q) \arrow[r, hook] \arrow[d] & M \arrow[d, "f"] \\
\{q\} \arrow[r, hook] & N
\end{tikzcd}$$

of C^∞ manifolds induces a commutative diagram of the corresponding tangent spaces,

$$\begin{tikzcd}
T_pf^{-1}(q) \arrow[r, hook] \arrow[d] & T_pM \arrow[d, "f'(p)"] \\
0 = T_qq \arrow[r] & T_qN
\end{tikzcd}$$

This shows that $T_pf^{-1}(q) \subseteq \ker(f'(p))$; none of the assumptions of the theorem were needed here. Now we look at the local picture to get the reverse inclusion.

By our earlier work, we can take the local picture to just be a surjective linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ (note that $L(0) = 0$). Then $L^{-1}(0)$ is already a linear subspace, and $T_0(L^{-1}(0)) = L^{-1}(0) = \ker(L)$ and $L'(0) = L$. Thus, the reverse inclusion is essentially a tautology in this case. ☐

Example. The Stiefel variety $V(n,k)$ is defined to be

$$V(n,k) = \{(v_1,\dots,v_k) \mid v_i \in \mathbb{R}^n,\ \langle v_i, v_j\rangle = \delta_{ij} \text{ for all } i,j\}.$$

For example, $V(n,1) = \mathbb{S}^{n-1}$. Considering the map $V(n,2) \to V(n,1)$ defined by $(v_1,v_2) \mapsto v_1$, we see that all fibers are diffeomorphic to $\mathbb{S}^{n-2}$. This is an example of a fiber bundle.

We also have $V(n,n) = \mathrm{O}(n)$, the orthogonal group. Given a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $\langle Lv, Lw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^n$, then $\langle Le_i, Le_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ for all i,j, where the e_i are the standard basis vectors. We can set $Le_i = v_i$.

▼ How does the rest of this identification proceed? ▼

We can identify $V(n,n-1) \cong \mathrm{SO}(n)$ as follows. We can define a map $F : (\mathbb{R}^n)^k \to \mathbb{R}^{k(k-1)/2}$ by $(v_1,\dots,v_n) \mapsto \langle v_i, v_j\rangle$ for $1 \le i \le j \le k$. Check that $F'(p)$ is onto for all $p \in V(n,k)$.

Definition. Let M, N be C^∞ manifolds, let $A \subseteq N$ be a C^∞ submanifold, and let $f : M \to N$ be a C^∞ map. We say that f is tranvserse to A at a point $p \in f^{-1}(A)$ when the composition

$$T_pM \xrightarrow{f'(p)} T_{f(p)}N \longrightarrow T_{f(p)}N/T_{f(p)}A$$

is surjective, or equivalently, if

$$T_{f(p)}A + f'(p)T_pM = T_{f(p)}N.$$

We then say that f is transverse to A when it is transverse to A at p for all $p \in f^{-1}(A)$.

Proposition. *If $f : M \to N$ is transverse to a C^∞ submanifold $A \subseteq N$, then*

(a) $f^{-1}(A)$ is a C^∞ submanifold, and

(b) for all $p \in f^{-1}(A)$, if $q = f(p)$, then $T_pM/T_pf^{-1}(A) \cong T_qN/T_qA$.

Assuming the first part of this proposition, the commutative diagram of manifolds

$$\begin{tikzcd}
f^{-1}(A) \arrow[r, hook] \arrow[d] & M \arrow[d, "f"] \\
A \arrow[r, hook] & N
\end{tikzcd}$$

induces a commutative diagram on tangent spaces, where the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & T_pf^{-1}(A) \arrow[r, hook] \arrow[d] & T_pM \arrow[r] \arrow[d, "f'(p)"] & T_pM/T_pf^{-1}(A) \arrow[r] \arrow[d, "\overline{f'(p)}"] & 0 \\
0 \arrow[r] & T_qA \arrow[r, hook] & T_qN \arrow[r] & T_qN/T_qA \arrow[r] & 0
\end{tikzcd}$$

The second part is asserting that the final vertical arrow, $\overline{f'(p)}$, is an isomorphism.

Proof of Proposition. We may assume that $N = \mathbb{R}^{k+\ell}$, and letting $x_1,\dots,x_{k+\ell}$ be the coordinate functions of N, we may assume that A is given by $(x_1,\dots,x_k) = (0,\dots,0)$.

Our map $f : M \to N \hookrightarrow \mathbb{R}^{k+\ell}$ has coordinate functions $(f_1, \dots, f_{k+\ell})$. Each of the f_i are C^∞ functions on M. Let $g : M \to \mathbb{R}^k$ be the map $g = (f_1, \dots, f_k)$. Note that, by the assumption that f is transverse to A, we have that $g'(p)$ is onto for all $p \in M$ such that $g(p) = 0$, i.e. all p that are in

$$g^{-1}(0) = \{\text{points where } f_1 = \cdots = f_k = 0\} = f^{-1}(A).$$

Thus, the implicit function theorem implies that $g^{-1}(g(p)) = f^{-1}(A)$ is smooth at p for all such p, i.e. that $f^{-1}(A)$ is a C^∞ submanifold of M. This proves the first part of the proposition. By the corollary to the implicit function theorem, we also obtain the identification

$$T_p f^{-1}(A) = T_p g^{-1}(0) = \ker(g'(p) : T_p M \to T_0 \mathbb{R}^k) = (f')^{-1}(T_q A)$$

which gives us the second part of the proposition. $\square$

Lecture 5 (2013-01-16)

Definition. Let P and Q be C^∞ submanifolds of a C^∞ manifold M. We say that P and Q intersect transversely if for all $x \in P \cap Q$, we have $T_xP + T_xQ = T_xM$. This guarantees that $P \cap Q$ is a submanifold, and that $T_x(P \cap Q) = T_xP \cap T_xQ$.

Let $i : P \hookrightarrow M$. Then to say that P meets Q transversely is equivalent to saying that the map i is transverse to Q.

If $x \in P$, then via $i'(x) : T_xP \to T_xM$ we can regard T_xP as a subspace of T_xM. Then the statement that i is transverse to Q is just saying that $i'(x)T_xP + T_xQ = T_xM$, and $i'(x)T_xP = T_xP$ and $T_x i^{-1}(Q) = i'(x)^{-1}T_xQ$, so that $T_x(P \cap Q) = T_xP \cap T_xQ$.

Usually, when people talk about submanifolds, e.g. "A is a smooth submanifold of M", they mean what we have discussed, but with the additional assumption that A is closed. From now on, we will adopt this convention, and when I want to refer to our earlier, more general notion, we will say "locally closed submanifold".

Additionally, all manifolds will now be assumed to be second-countable and Hausdorff.

Some good sources the topics we'll be discussing are Milnor's *Topology from the Differential Viewpoint* and Guillemin and Pollack's *Differential Topology.*

Our goal will be to put an equivalence relation on the collection of all closed submanifolds of codimension r of a C^∞ manifold y, called concordance.

Theorem (Transversality theorem)**.** *If X and Y are compact C^∞ manifolds, $A \hookrightarrow Y$ is a C^∞ submanifold, and $f_0 : X \to Y$ is a C^∞ map, then there is a C^∞ map $f : X \to Y$ which is transverse to A and which is homotopic to f_0 (you may also see it stated as $\|f - f_0\| < \epsilon$).*

Theorem (Sard's theorem)**.** *If X and Y are C^∞ manifolds and $f : X \to Y$ is a C^∞ map, then*

$$\{y \in Y \mid y \textit{ is not a regular value of } f\}$$

is a set of measure zero.

When we get to proving Sard's theorem, we'll say what we mean by a set of measure zero.

Definition. Given C^∞ manifolds M and N, there is a unique C^∞ manifold structure on $M \times N$ such that, if $p : M \times N \to M$ and $q : M \times N \to N$ are the projection maps and $f : X \to M \times N$ is some map, then X is C^∞ if and only if $p \circ f$ and $q \circ f$ are both C^∞.

Definition. A deformation of a C^∞ map $f_0 : X \to Y$ is a C^∞ map $F : X \times S \to Y$, where $S \subset \mathbb{R}$ is an open neighborhood of 0, such that $F(x, 0) = f_0(x)$ for all $x \in X$. We say that a deformation is rich if the derivative $F'(x, 0) : T_{(x,0)}(X \times S) \to T_{f_0(x)}Y$ is surjective for all $x \in X$.

Note that, given a deformation F of f_0, we get a C^∞ map $f_s : X \to Y$ for all $s \in S$, namely $f_s(x) = F(x, s)$. If S is connected, then f_s is homotopic to f_0 for all $s \in S$.

For example, if $Y = \mathbb{R}$, we can take $S = \mathbb{R}$, and define $F : X \times S \to Y$ by $F(x, s) = f_s(x) = f_0(x) + s$. This deformation is rich because, for any $x \in X$,

$$\frac{d}{dt}(f_0(x) + ts)\bigg|_{t=0} = s.$$

Lemma 1. *If $F : X \times S \to Y$ is a rich deformation and X is compact, then there is a open neighborhood $S' \subset S$ of 0 such that $F|_{X\times S'}$ induces surjections on all tangent spaces.*

Proof. Either by the implicit function theorem, or by noting that

$$\{T \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n) \mid T \text{ is surjective}\}$$

is open, we see that

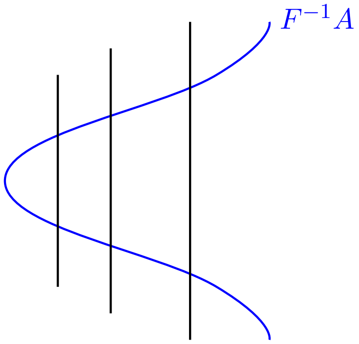
$$U = \{(x, s) \in X \times S \mid F'(x, s) \text{ is onto}\}$$

is an open set in $X \times S$, and $X \times 0 \subset U$. Now, the fact that X is compact implies that there is some neighborhood S' of 0 in S such that $X \times S' \subset U$. ☐

Proof of transversality theorem. We may as well replace S by S', so we assume that $F'(x, s)$ is surjective for all $(x, s) \in X \times S$. We see that F is transverse to A, because a composite of surjections is a surjection:

$$T_{(x,s)}(X \times S) \xrightarrow{F'(x,s)} \!\!\!\!\rightarrow T_yY \longrightarrow\!\!\!\!\rightarrow T_yY/T_yA.$$

Therefore, $F^{-1}A$ is a C^∞ submanifold. For example, if $X = Y = \mathbb{R}$, then the picture in $X \times S$ might look like



where the lines represent the fibers over different $s \in S$, and the points where the fiber over s intersects $F^{-1}A$ simply constitute $f_s^{-1}(A)$.

▼ Did we choose a specific s, and if so, how? ▼

$$\begin{tikzcd}
F^{-1}A \arrow[r, hook] \arrow[dr, "q"'] & X \times S \arrow[d, "p_2"] \\
& S
\end{tikzcd}$$

We want to show that s is a regular value of q; you should do this yourselves. Now, once we prove the following lemma, we are done. ☐

Lemma 2. *As above, let $q : F^{-1}A \to S$ be the restriction of p_2 to $F^{-1}A$. Then $s \in S$ is a regular value of q if and only if $f_s : X \to Y$ is transverse to A.*

Note that Sard's theorem implies the existence of such points $s \in S$.

▼ Missing content ▼

Proof of Lemma 2. Let $(x,s) \in F^{-1}A$, and $y = F(x,s) \in A$.

$$\begin{tikzcd}
 & & 0 \\
 & T_xX \oplus T_sS \arrow[d, equal] & T_yY/T_yA \arrow[u] \\
 & T_{(x,s)}(X \times S) \arrow[r, "F'(x,s)", "\text{onto}"'] & T_yY \arrow[u] \\
F'(x,s)T_yA \arrow[r, phantom, "="] & T_{(x,s)}F^{-1}A \arrow[u, hook] \arrow[r] & T_yA \arrow[u, hook] \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

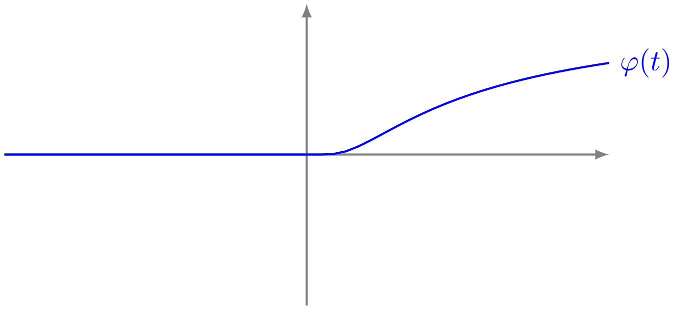
Then $F'(x,s)(v,0) = f_0'(x)v$ for all $v \in T_xX$. ☐

Lemma. *Every compact C^∞ manifold Y is diffeomorphic to a submanifold of $\mathbb{R}^n$.*

Proof. Define the function φ by

$$\varphi(t) = \begin{cases} e^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

This is a C^∞ function whose graph looks like



Given $a < b \in \mathbb{R}$, define $f(x) = (b-x)(x-a)$. Now define $g(x) = \int_{-\infty}^x \varphi(f(t))\,dt$. Since the function $\varphi(f(t))$ is compactly supported its total integral is finite, so we can normalize the function g so that its integral is 1.

Ultimately, we can make a C^∞ function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\psi(x) = 1$ if $\|x\| \leq \alpha$ and $\psi(x) = 0$ if $\|x\| \geq \beta$ for any choice of $0 < \alpha < \beta$. Then consider the function $M : \mathbb{R}^n \to \mathbb{R}^n$ defined by $x \mapsto \psi(x)x$. Note that $M'(x)$ is injective on $T_x\mathbb{R}^n$ for $\|x\| < \alpha$, and is 0 if $\|x\| \geq \beta$.

▼ Missing content ▼

If Y is a compact Hausdorff manifold, we have plenty of open sets $U \subset Y$.

$$U \xrightarrow[\text{diff}]{h} \Omega = B(0; 2\beta) \xrightarrow{M} \mathbb{R}^m$$

M extends to a C^∞ function on Y. $\square$

Lecture 6 (2013-01-18)

There was a mistake in the statement of exercise 5.10.

One way of producing deformations of $f_0 : X \to Y$ is via the tubular neighborhood theorem. This is an example of where we'll use existence and uniqueness of solutions to ODE's.

There are many forms of the theorem, but we'll state it for Y compact.

Theorem (Tubular neighborhood theorem). *Let $Y \subseteq \mathbb{R}^{m+n}$ be a compact C^∞ submanifold of dimension n. Let*

$$N = \{(y, v) \in Y \times \mathbb{R}^{m+n} \mid \langle w, v\rangle = 0 \textit{ for all } w \in T_yY\},$$

and let

$$N_\epsilon = \{(y, v) \in N \mid \|v\| < \epsilon\}.$$

Then

1. *N is a closed C^∞ submanifold of $Y \times \mathbb{R}^{m+n}$.*
2. *There is some $\epsilon > 0$ such that $(y, v) \mapsto y + v$ for $(y, v) \in N_\epsilon$ gives a diffeomorphism from N_ϵ to an open subset of $\mathbb{R}^{m+n}$. We say that this open subset is a "tubular neighborhood".*

Proof. Because we can check the condition of being a submanifold locally, to show that N is a submanifold it suffices to cover Y by open subsets U such that $p_1^{-1}(U)$ is a C^∞ submanifold of $U \times \mathbb{R}^{m+n}$, where $p_1 : N \to Y$ is the projection map. We may assume that each U is of the form $\{(x, f(x)) \mid x \in \Omega\}$ for some open $\Omega \subseteq \mathbb{R}^n$ and for some C^∞ map $f : \Omega \to \mathbb{R}^m$ (after some permutation in S_{n+m}).

Let $y = (x, f(x))$, with $x \in \Omega$. Then

$$T_yY = \{(y, f'(x)v) \mid v \in \mathbb{R}^n\}.$$

Write $w = (w_1, w_2)$ where $w_1 \in \mathbb{R}^n$ and $w_2 \in \mathbb{R}^m$. Then

$$\langle v, w_1\rangle + \langle f'(x)v, w_2\rangle = \langle v, w_1\rangle + \langle v, f'(x)^* w_2\rangle = 0$$

where $f'(x)^* : \mathbb{R}^m \to \mathbb{R}^n$ is the adjoint of $f'(x) : \mathbb{R}^n \to \mathbb{R}^m$.

For all $v \in \mathbb{R}^n$, $\langle v, w_1 + f'(x)^* w_2\rangle = 0$, so $w_1 = -f(x)^* w_2$. Thus

$$p_1^{-1}(U) = \{\underbrace{(x, f(x))}_{\in Y}, (-f'(x)^* w_2, w_2) \mid x \in \Omega,\ w_2 \in \mathbb{R}^m\}.$$

If $\varphi : M \to \mathbb{R}^n$ is C^∞ then its graph in $M \times \mathbb{R}^n$ is a C^∞ submanifold (with $M = U \times \mathbb{R}^m$, we get the result).

Now to part 2. Let $A : N \to \mathbb{R}^{m+n}$ be defined by $A(y, v) = y + v$ for all $(y, v) \in N$. Note that we have $p^{-1}(U) \cong U \times \mathbb{R}^m$.

Let's discuss $T_{(y_0,0)}N$. We have two submanifolds of N, namely $N_{y_0} = p_1^{-1}(y_0)$ and $Y \hookrightarrow N$ (this is just $y \mapsto (y, 0)$).

We see that N_{y_0} and Y are submanifolds of N that meet transversely at $(y_0, 0)$, and in fact

$$T_{y_0}Y \oplus T_{(y_0,0)}(N_{y_0}) = T_{(y_0,0)}N.$$

By definition, $A'(y_0,0)(\alpha,0)=\alpha$ for all $\alpha \in T_{y_0}Y$, and $A'(y_0,0)(0,w)=(0,w)$. Therefore $A'(y_0,w)=y_0+w$. It follows that $A'(y_0,0): T_{(y_0,0)}N \xrightarrow{\cong} T_{y_0}\mathbb{R}^{n+m}$.

Then $Z=\{(y,v) \mid A'(y,v) \text{ is a diffeomorphism}\}$ is an open subset $\supseteq Y\times 0$. The compactness of Y implies that there is some $\epsilon>0$ such that $N_\epsilon \subseteq Z$.

Now, we claim that there is some $\epsilon>0$ such that $A|_{N_\epsilon}$ is injective. If not, we'd get $(y_n', v_n') \neq (y_n'', v_n'')$, both in $N_{1/n}$, such that $y_n'+v_n' = y_n''+v_n''$ $(*)$. Since Y is compact, we may assume that $\lim_{n\to\infty} y_n' =: y'$ and $\lim_{n\to\infty} y_n'' =: y''$ exist. We now get $y'=y''$, by taking the limit of $*$ as $n\to\infty$.

Both sequences eventually lie in a fixed neighborhood of $(y',0)\in N$, but this contradicts the inverse function theorem because $A'(y,0)$ is a diffeomorphism.

Thus $A|_{N_\epsilon}: N_\epsilon \to \mathbb{R}^{n+m}$ is injective, open, and A' is injective for all points of N_ϵ. Thus, $A(N_\epsilon)=U_\epsilon$ is open, and $A: N_\epsilon \to U_\epsilon$ is a diffeomorphism to an open subset $U_\epsilon \subset \mathbb{R}^{m+n}$. □

Now, let's prove the existence of rich deformations of $f_0 : X \to Y$ where both are compact C^∞ manifolds.

Proof. Let $i: Y \hookrightarrow \mathbb{R}^{m+n}$. We have $Y \subset U_\epsilon \subset \mathbb{R}^{m+n}$.

We know that $i \circ f_0$ has a rich deformation $G: X\times S \to \mathbb{R}^{m+n}$, i.e. $G'(x,0)$ is surjective for all $x\in X$, and $G(x,0)=f_0(x)$ for all $x\in X$. Then $G^{-1}(U_\epsilon)$ is open in $X\times S$, which contains $X\times\{0\}$.

Because X is compact, there is an open neighborhood $S'\subseteq S$ of 0 such that $X\times S' \subseteq G^{-1}(U_\epsilon)$. Thus, $G(X\times S')\subseteq U_\epsilon$.

There is a C^∞ retraction $r: U_\epsilon \to Y$ (remember that U_ϵ is a "tube" around Y).

$$\begin{tikzcd}
X\times S' \arrow[r, "G"] \arrow[rr, bend right, "F"'] & U_\epsilon \arrow[r, "r"] & Y
\end{tikzcd}$$

$$\underbrace{p_1(A|_{N_\epsilon})^{-1}}_{=r} y = y \text{ for all } y\in Y$$

$$\begin{tikzcd}
N_\epsilon \arrow[r, "A|_{N_\epsilon}", "\cong"'] \arrow[d, "p_1"'] & U_\epsilon & Y \arrow[l, hook'] \\
Y & Z &
\end{tikzcd}$$

For all $x\in X$, $f_0(x)\in Y$, and

$$F(x,0) = rG(x,0) = rf_0(x) = f_0(x).$$

$G'(x,0)$ is onto by assumption for all $x\in X$.

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & U_\epsilon \arrow[r, "r"] & Y
\end{tikzcd}$$

so $r'(x)$ is onto for all $x \in Y$.

$G(x,0)\in Y$ so $r'(f_0(x))\cdot G'(x,0) = F'(x,0)$ is onto. □

Let X be a compact C^∞ manifold. We want a "rich" (our definition of this can vary) deformation of $f_0 : X \to Y$ such that $f_s : X \to Y$ is an embedding.

Let $F : X \times S \to Y$ be a deformation of f_0.

$$B_0 = \{(x_1, x_2) \in X \times X \mid x_1 \neq x_2 \text{ and } f_0(x_1) = f_0(x_2)\}$$

$$B = \{(x_1, x_2, s) \in X \times X \times S \mid x_1 \neq x_2 \text{ and } F(x_1, s) = F(x_2, s)\} \xrightarrow{q} S$$

$q^{-1}s$ is the "bad set of $f_s : X \to Y$".

$$\begin{tikzcd} X \times X \times S \arrow[r, "G"] & Y \times Y \\ (x_1, x_2, s) \arrow[r, mapsto] & (F(x_1, s), F(x_2, s)) \end{tikzcd}$$

Then $G^{-1}(\Delta Y) = B$, where ΔY is the image of the diagonal embedding $Y \to Y \times Y$. We may **hope** that G is transverse to ΔY. Note that ΔY is C^∞ submanifold whose codimension in $Y \times Y$ is $\dim(Y)$. $B \subset X \times X \times S$ is a submanifold of codimension $= \dim(Y)$, i.e. $\dim(B) = (2\dim(X) - \dim(Y)) + \dim(S)$.

Consider $q : B \to S$. Sard's theorem say that there are plenty of regular values of q. If s is a regular value, then B_s is a C^∞ submanifold of B of codimension $= \dim(S)$.

$$\dim(B_s) = \dim(B) - \dim(S) = 2\dim(X) - \dim(Y) + \dim(S) - \dim(S).$$

Thus, under the assumption that $2\dim(X) < \dim(Y)$, we may hope that $f_s : X \to Y$ is injective. To get an embedding, we have to work a little more.

Lecture 7 (2013-01-23)

Theorem (Whitney embedding theorem)**.** *Let X and Y be C^∞ manifolds, and assume that X is compact. If $2\dim(X)+1 \leq \dim(Y)$, then every $f_0 : X \to Y$ can be approximated by an embedding.*

Proof. We will give a proof in the case when Y is $\mathbb{R}^m$, and using tubular neighborhoods it will work even when Y is compact.

Let's write $Y = V$ where V is a vector space of dimension m (a notational aid for when we are using the vector space structure). Let $g_1, \dots, g_r$ be C^∞ functions from X to $\mathbb{R}$, and let $v_1, \dots, v_m$ be a basis for V. Define a deformation $F : X \times S \to Y$ of f_0, where $S = \mathbb{R}^{rm}$, by

$$F(p,t) = f_0(p) + \sum_{i,j} t_{ij} g_i(p) v_j,$$

where $t = (t_{ij}) \in S$.

Define $G : (X \times X \setminus \Delta X) \times S \to V \times V$ by $G(p,q,t) = (F(p,t), F(q,t))$. We want G to be transverse to $\Delta V \subset V \times V$.

Let $H : V \times V \to V$ be defined by $H(v_1, v_2) = v_1 - v_2$. Then $H'(v_1, v_2)$ is onto, 0 is a regular value of H, and $H^{-1}(0) = \Delta V$.

We see that we want to demonstrate the surjectivity of $(H \circ G)'(p,q,t)$ at all points in

$$\{(p,q,t) \mid p \neq q \text{ and } F(p,t) = F(q,t)\}.$$

Now note that

$$(H \circ G)(p,q,t) = f_0(p) - f_0(q) + \sum_{i,j} t_{ij}(g_i(p) - g_i(q))v_j.$$

Let's look at the derivative evaluated at points of the form $(0,0,?)$, as it is easiest to compute in this case, and it will suffice for surjectivity anyway.

Because the v_j form a basis for V, a necessary and sufficient condition that

$$\begin{tikzcd} TS \arrow[r] & V \\ (0,0,?) \arrow[r, mapsto] & (H \circ G)'(p,q,t)(0,0,?) \end{tikzcd}$$

is surjective is that there is some i such that $g_i(p) \neq g_i(q)$.

For instance, you could consider an embedding

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "g_i"'] & \mathbb{R}^N \arrow[d, "p_i"] \\ & \mathbb{R} \end{tikzcd}$$

where $N = r$.

Now, $G^{-1}(\Delta V)$ is a C^∞ submanifold of $(X \times X \setminus \Delta X) \times S$.

Let $Q : G^{-1}(\Delta V) \to S$ be the projection.

Consider the regular values of Q (which are a subset of S). For any regular value s, we have

$$Q^{-1}(s) = \{(p,q) \in X \times X \mid p \neq q \text{ and } f_s(p) = f_s(q)\}$$

is a C^∞ submanifold of $X \times X \setminus \Delta X$ of dimension $2\dim(X) - \dim(Y)$. Thus, if $2\dim(X) < Y$, this set is empty.

For $f : X \to Y$ to be an embedding, we need that

(i) f is injective,

(ii) $f'(p)$ is injective for all $p \in X$,

(iii) $f|_X : X \to f(X)$ is a homeomorphism and $f(X)$ is a closed subset of Y,

though if X is compact then (iii) is not required.

The tangent bundle

The tangent bundle of X is denoted by TX. As a set, it is equal to the disjoint union

$$TX = \coprod_{x \in X} T_x X.$$

Given an open $\Omega \subseteq \mathbb{R}^n$, there is a natural identification $T\Omega \cong \Omega \times \mathbb{R}^n$ (this is the topology and C^∞ structure on $T\Omega$).

In general, the topology on TX is defined as follows. There is a natural projection $\pi : TX \to X$ defined by $\pi(T_x X) = x$ for all $x \in X$.

If $f : P \to Q$ is C^∞, then there is an induced commutative diagram

$$\begin{tikzcd}
TP \arrow[r, "F"] \arrow[d, "\pi"'] & TQ \arrow[d, "\pi"] \\
P \arrow[r, "f"'] & Q
\end{tikzcd}$$

where F is the map defined by $F|_{T_x P} = f'(x)$ for all $x \in P$.

Now let X be a C^∞ manifold. For all charts $(\Omega, g : \Omega \to U)$ on X, i.e. with $g : \Omega \to U$ a diffeomorphism from an open $\Omega \subset \mathbb{R}^n$ to an open $U \subseteq X$, we get a commutative diagram

$$\begin{tikzcd}
T\Omega \arrow[r, "\text{homeo}"] \arrow[d] \arrow[rr, bend left, "h"] & TU \arrow[r, hook] \arrow[d] & TX \arrow[d] \\
\Omega \arrow[r, "g"'] & U \arrow[r, hook] & X
\end{tikzcd}$$

The C^∞ structure is transported via h.

Returning to our argument, we have $F : X \times S \to V$, a deformation of f_0. This gives rise to $\widetilde{F} : (TX) \times S \to TV$, a C^∞ map.

Let $w \in T_p X$ be such that $\widetilde{F}(w, s) = F'(p, s)(w, 0)$.

Now define $T'X$ to be the complement of the zero section of TX; in other words,

$$T'X = \coprod_{x \in X} (T_xX \setminus \{0\}).$$

Clearly, $T'X$ is an open subset of TX.

We claim that $\widetilde{F} : (T'X) \times S \to V$ has $0 \in V$ as a regular value.

Let's assume the claim for now. Then $M := \widetilde{F}^{-1}(0)$ is a C^∞ submanifold with

$$2\dim(X) - \dim(S) - \dim(\widetilde{F}^{-1}(0)) = \dim(V) = \dim(Y).$$

Consider the map $Q' : M \to S$, and let s be a regular value of Q'. Then

$$\dim((Q')^{-1}(s)) = \dim(M) - \dim(S) = 2\dim(X) - \dim(Y).$$

Under the assumption that $2\dim(X) \leq \dim(Y)$, the fiber is forced to be empty,

$$(Q')^{-1}(s) = \{(p, w) \in TX \mid 0 \neq w \in T_pX \text{ for } p \in X, \text{ and } f_s'(p)w = 0\}.$$

$$\boxed{\text{If } 2\dim(X) \leq \dim(Y), \text{ then } f_s'(p) : T_pX \to T_{f_s(p)}V \text{ is injective for all } p \in X.}$$

Proof of claim. Fix $p \in X$ and $0 \neq w \in T_pX$. We have the map $\widetilde{F}|(p, w) \times S \to V$. We need to find an i such that $g_i'(p)w \neq 0$.

Taking the derivative of $H \circ G$, we get

$$f_0'(p)w + \sum_{i,j} t_{ij}(g_i'(p)w)v_j.$$

In our case, j is a C^∞ embedding

$$\begin{tikzcd} X \arrow[r, hook, "j"] & \mathbb{R}^N \arrow[d, "p_i"] \\ & \mathbb{R} \end{tikzcd}$$

$0 \neq w \in T_pX$ implies that $j'(p)w \neq 0$, which implies that there is some i such that $(p_i \circ j)'(p)w \neq 0$, i.e. $g_i'(p) \neq 0$. ☐

Thus, we are done. ☐

Our proof shows a form of the Whitney immersion theorem (at least when $Y = \mathbb{R}^n$), then every $f_0 : X \to Y$ can be approximated by $f : X \to Y$ which is an immersion (i.e. $f'(p) : T_pX \to T_{f(p)}Y$ is injective for all $p \in X$) if $2\dim(X) \leq \dim(Y)$.

Next time we'll do Sard's theorem and maybe a bit more about vector bundles.

Corollary (of Sard's theorem)**.** *Let M and N be C^∞ smooth manifolds (we also assume second countable and T_2), let $f : M \to N$ be a C^∞ map, and assume that* $\dim(M) < \dim(N)$. *Then f is **not** onto. If $a \in N$ is a regular value of f, then $f^{-1}(a)$ manifold of dimension* $\dim(M) - \dim(N)$.

Lecture 8 (2013-01-25)

Today we'll talk about Sard's theorem.

All manifolds today will be second-countable and Hausdorff.

Definition. Given C^∞ manifolds M and N, and a C^∞ map $f : M \to N$, we define

$$\mathrm{Crit}(f) = \{x \in M \mid f'(x) \text{ is not surjective}\}.$$

Thus, $f(\mathrm{Crit}(f)) \subset N$. The complement $f(\mathrm{Crit}(f))^{\mathsf{c}}$ is called the set of regular values of f.

Theorem (Sard's theorem)**.** *The set $f(\mathrm{Crit}(f))$ has measure zero.*

Preliminaries

Let M be a C^∞ manifold, and $E \subseteq M$. We say that the measure of E is zero when, for all diffeomorphisms

$$\begin{tikzcd}
\Omega \arrow[r, "\varphi"] \arrow[d, hook, "\text{open}"'] & U \arrow[d, hook, "\text{open}"] \\
\mathbb{R}^n & M
\end{tikzcd}$$

we have $m^*(\varphi^{-1}(E)) = 0$, where m^* denotes outer Lebesgue measure.

Lemma. *Given an open $\Omega \subseteq \mathbb{R}^n$, a subset $E \subseteq \Omega$, and a C^1 map $f : \Omega \to \mathbb{R}^k$, then*

(a) if $k = n$ and $m^*(E) = 0$, then $m^*(f(E)) = 0$, and

(b) if $k > n$, then $m^*(f(\Omega)) = 0$.

Proof. For (a), it suffices to prove that $m^*(f(E \cap K)) = 0$ where K is compact. Given $x, y \in \Omega$ such that the line between them is contained in Ω, i.e. such that $tx + (1-t)y \in \Omega$ for all $0 \leq t \leq 1$, then

$$\|f(y) - f(x)\| \leq \sup\{\|f'(z)\| \mid z = tx + (1-t)y,\ 0 \leq t \leq 1\} \cdot \|y - x\|.$$

Given a compact $K \subset \Omega$, there is a $\delta > 0$ such that

$$K_\delta := \{x + v \mid x \in K,\ \|v\| \leq \delta\} \subseteq \Omega.$$

If $m^*(E \cap K) = 0$, then for all $\epsilon > 0$, there is some covering

$$E \cap K \subseteq \bigcup_{i=1}^\infty C_i$$

where the C_i are translates of $[-a_i, a_i]^n$ with $\mathrm{diam}(C_i) < \delta$ and $\sum m^*(C_i) < \epsilon$.

We have $\mathrm{diam}(f(C_i)) \leq S \,\mathrm{diam}(C_i)$ where $S = \sup\{\|f'(z)\| \mid z \in K_\delta\}$.

There is a fixed ratio S' between the volume of a ball and the volume of a cube.

Thus, $f(C_i) \subseteq$ an open ball of volume $\leq SS' \,\mathrm{vol}(C_i)$.

To prove (b), note that the claim reduces to considering mapping a cube C in n variables into an open ball B in k variables. Because we have

$$\frac{\mathrm{vol}(B)}{\mathrm{vol}(C)} < \text{constant} \cdot (\mathrm{diam}(C))^{k-n},$$

by covering a compact set K with cubes of sufficiently small diameter we are done. ☐

Corollary. *Given an open $\Omega \subseteq \mathbb{R}^n$ and open $U \subseteq M$, and a diffeomorphism $\varphi : \Omega \to U$, then if $E \subseteq \Omega$ has $m^*(E) = 0$, then $\varphi(E)$ has measure zero.*

Remark 1. To prove Sard's theorem, it suffices to consider the case when the domain is an open subset of $\mathbb{R}^n$ and the codomain is $\mathbb{R}^k$. This is because, given our map $f : M \to N$, we can cover N by countably many open sets $N = \bigcup_{m=1}^{\infty} U_m$, so that $M = \bigcup_{m=1}^{\infty} f^{-1}(U_m)$, and we can cover each $f^{-1}(U_m)$ by subsets diffeomorphic to open subsets of $\mathbb{R}^n$. Then note that the union of the $\mathrm{Crit}(f^{-1}(U_m) \to U_m)$ is $\mathrm{Crit}(f)$.

Remark 2. Part (b) of the lemma implies Sard's theorem in the case that dim(domain) < dim(codomain).

Proof of Sard's theorem.

Step 1

We take $\Omega \subseteq \mathbb{R}^n$ an open set and a C^∞ map $f : \Omega \to \mathbb{R}$.

The case of $n = 0$ is evident. We proceed by induction on $n > 0$. Note that

$$X_1 := \mathrm{Crit}(f) = \{x \in \Omega \mid \partial_i f(x) = 0 \text{ for all } 1 \le i \le n\},$$

and we can form a decreasing sequence of closed sets

$$X_k := \{x \in \Omega \mid \text{all partial derivatives of order } r \text{ vanish, for all } 1 \le r \le k\}.$$

It will suffice to show that

(a) $f(X_m - X_{m+1})$ has measure zero for all $m = 1, 2, 3, \ldots$, and

(b) $f(X_\infty)$ has measure zero, where $X_\infty = X_1 \cap X_2 \cap X_3 \cap \cdots$.

For each multi-index $(i_1, \ldots, i_{m+1})$, consider the set

$$Y = \{x \in \Omega \mid \partial_{i_1}\partial_{i_2} \cdots \partial_{i_{m+1}} f(x) \ne 0 \text{ and } \underbrace{\partial_{i_2} \cdots \partial_{i_{m+1}} f(x)}_{:= \varphi(x)} = 0\}.$$

We have a finite collection of such Y's, and $X_m - X_{m+1}$ is contained in the union of these Y. It suffices to show that $f(Y \cap X_1)$ has measure zero for each such Y. Rewrite Y as

$$Y = \{x \in \Omega \mid \varphi(x) = 0 \text{ and } \tfrac{\partial \varphi}{\partial x_{i_1}}(x) \ne 0\}.$$

Y is a C^∞ submanifold of dimension $n - 1$ of Ω by _____. We have

$$\begin{aligned} X_1 \cap Y &= \{x \in Y \mid f'(x) : T_x\Omega \to \mathbb{R} \text{ is zero}\} \\ &= \{x \in Y \mid f'(x)|_{T_xY} : T_xY \to \mathbb{R} \text{ is zero}\} \end{aligned}$$

$$= \mathrm{Crit}(f|_Y)$$

It follows that $f(Y \cap X_1) \subseteq f(\mathrm{Crit}(f|_Y))$ which has measure zero by induction hypothesis.

Now we turn to part (b). We have

$$X_\infty = \{p \in \Omega \mid \text{the Taylor expansion of } f - f(p) \text{ is zero at } p\}.$$

Fix a compact set $K \subset \Omega$, and points $a, x \in K$ such that the line segment joining them is contained inside K. Let $a \in X$

Cover $X_\infty \cap K$ by cubes of diameter δ, such that the sum of the volume of the cubes is $\leq D$. The key fact is that there is some constant L such that

$$\mathrm{diam}(f(C)) \leq L \cdot \mathrm{diam}(C)^{n+1},$$

which follows from the inequality $|f(x) - f(a)| \leq C\|x - a\|^{n+1}$. We have

$$\begin{aligned} m^*(\textstyle\bigcup f(C)) &\leq L \sum \mathrm{diam}(C)^{n+1} \\ &\leq \mathrm{diam}(C) \cdot L \sum \mathrm{diam}(C)^n \\ &\leq \mathrm{diam}(C) \cdot LL'D \end{aligned}$$

and $\mathrm{diam}(C)$ can be shrunk to zero, so we are done.

Step 2

We have proven Sard's theorem for a map $f : \Omega \to \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^n$ is open. Now we need to prove the claim for maps $f : \Omega \to \mathbb{R}^k$ for all $k \geq 1$. We proceed by induction on k, because we have just dealt with the case of $k = 1$.

▼ I missed the details here / possible mistakes ▼

Consider the projection

$$\begin{tikzcd} \Omega \arrow[r, "f"] \arrow[dr, "\pi\circ f"'] & \mathbb{R}^k \arrow[d, "\pi"] \\ & \mathbb{R}^{k-1} \end{tikzcd}$$

where $\pi(x_1, \ldots, x_k) = (x_1, \ldots, x_{k-1})$. We clearly have that $\mathrm{Crit}(\pi \circ f) \subseteq \mathrm{Crit}(f)$. Note that if $E \subseteq \mathbb{R}^{k-1}$ has $m^*(E) = 0$, then we have $m^*(E \times \mathbb{R}) = 0$. Thus, it suffices to show that

$$f(\{x \in \Omega \mid f'(x) \text{ is not onto but } (\pi \circ f)'(x) \text{ is onto}\})$$

is of measure zero. If $(\pi \circ f)'(x)$ is onto, we can choose coordinates on Ω such that

$$(\pi \circ f)(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}),$$

i.e. we can assume that $f(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, \varphi(x_1, \ldots, x_n))$. We have submersions π and π' such that

$$\begin{tikzcd}
\Omega \arrow[r, "f"] \arrow[dr, "\pi'"'] & \mathbb{R}^k \arrow[d, "\pi"] \\
& \mathbb{R}^{k-1}
\end{tikzcd}$$

Then $x \in \operatorname{Crit}(f)$ if and only if $x \in \operatorname{Crit}(f|_{(\pi')^{-1}\pi'(x) \to \pi^{-1}\pi'(x)})$.

We've essentially frozen $(x_1, \ldots, x_{k-1})$. Then $\frac{\partial \varphi}{\partial x_i} = 0$ for all $i = k, \ldots, n$.

$$\begin{aligned} E &= f(\operatorname{Crit}(f)) \cap (\{a\} \times \mathbb{R}) \\ &= \{\tfrac{\partial \varphi}{\partial x_i}(a, x_k, \ldots, x_n) = 0 \text{ for all } i\} \end{aligned}$$

where $a = (a_1, \ldots, a_{k-1})$.

The set $E \cap \pi^{-1}(a)$ is measure zero for all a.

We'd like to use Fubini's theorem to finish, but the problem is that Fubini's theorem requires that the function be measurable to start with, which we can't guarantee. Thus, we'll have to use the same sort of estimates we've been using to produce a correct proof. You can find the details in Milnor's book. □

Lecture 9 (2013-01-29)

This week, we'll cover the connection with algebraic topology.

Theorem. *If X is a compact connected (non-empty) orientable n-manifold, then*

(a) $H_n(X) \cong \mathbb{Z}$,

(b) for all $x \in X$, there is a natural map $H_n(X) \xrightarrow{\cong} H_n(X, X - x) \cong \mathbb{Z}$.

Definition. An orientation of X is a generator of $H_n(X)$, and an oriented manifold is a pair (X, θ) where θ is an orientation of X.

Theorem. *If we remove the hypothesis that X is orientable, then it is still true that*

(a′) $H_n(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$,

(b′) there is a natural map $H_n(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H_n(X, X - x; \mathbb{Z}/2\mathbb{Z})$.

Definition. Let $f : X \to Y$ be continuous, where both (X, θ_X) and (Y, θ_Y) are compact connected oriented n-manifolds. The degree of f is the unique integer $\deg(f)$ such that $f_*(\theta_X) = \deg(f) \cdot \theta_Y$ where $f_* : H_n(X) \to H_n(Y)$ is the induced map on homology. This does not depend on the choice of θ_X or θ_Y.

When X and Y are not assumed to be orientable, we can still define the degree of f in $\mathbb{Z}/2\mathbb{Z}$.

Definition. Let X and Y be oriented n-manifolds, and let $x \in X$ be an isolated point of $f^{-1}(f(x))$. Let $y = f(x)$. Thus, we have a neighborhood $U \subseteq X$ of x such that $f^{-1}(f(x)) \cap U = \{x\}$. In fact, we can get a neighborhood $U \subseteq X$ of x and $U' \subseteq Y$ of y such that $f(U) \subseteq U'$ and $f(U \setminus \{x\}) \subseteq U' \setminus \{y\}$. The map f induces

$$\begin{tikzcd}
 & H_n(U, U-x) \arrow[r, "``f_*\text{''}"] \arrow[d, "\cong"', "\text{excision}"] & H_n(U', U'-y) \arrow[d, "\cong"', "\text{excision}"] & \\
H_n(X) \arrow[r, "\cong"] & H_n(X, X-x) & H_n(Y, Y-y) & H_n(Y) \arrow[l, "\cong"']
\end{tikzcd}$$

$$\theta_X \in H_n(X) \longmapsto \theta_X(x) \in H_n(U, U-x) \longmapsto \theta_Y(y) \in H_n(U', U'-y) \longmapsto \theta_Y \in H_n(Y)$$

We then define the local degree of f at x to be the unique integer $\deg_x(f)$ such that

$$\deg_x(f) \cdot \theta_Y(y) = ``f_*\text{''}\theta_X(x).$$

Theorem. *Let X and Y both be compact connected oriented (non-empty) n-manifolds, and let $f : X \to Y$ be continuous. Let $y \in Y$, and assume that $f^{-1}(y)$ is finite. Then*

$$\deg(f) = \sum_{x \in f^{-1}(y)} \deg_x(f).$$

Proof. We will use singular homology. We see that there is a neighborhood U_y' of y in Y such that $f^{-1}(U_y) = \coprod_{x \in f^{-1}(y)} U_x$ where U_x is a neighborhood of x. We get an open cover of X, consisting of

$$\{U_x \mid x \in f^{-1}(y)\} \cup \{X \setminus f^{-1}(y)\}$$

▼ Why is this true? ▼

The homology class θ_X is representable by a cycle

$$\xi = \sum_{x \in f^{-1}(y)} \sigma_x + \eta,$$

where $\sigma_x \in C_n(U_x)$ and $\eta \in C_n(X - f^{-1}(y))$.

Because ξ is a cycle, we have $\partial\xi = 0 \in C_{n-1}(X)$. Thus, $f_*(\theta_X)$ is represented by

$$\begin{tikzcd}
\sum f(\sigma_x) + f(\eta) \in Z_n(Y) \arrow[r, hook] \arrow[d] & C_n(Y) \arrow[d] \\
Z_n(Y, Y-y) \arrow[r, hook] & C_n(Y, Y-y)
\end{tikzcd}$$

We have $\eta \in C_n(X - f^{-1}(y))$, and $f(X - f^{-1}(y)) \subseteq Y - \{y\}$, so $f(\eta) \in C_n(Y - y)$. Thus, when we send $\theta_X \in H_n(X)$ via

$$H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{\cong} H_n(Y, Y-y)$$

the image can be represented by $\sum_{x \in f^{-1}(y)} f(\sigma_x)$, because $f(\eta)$ is supported on $Y - y$.

▼ Missing content ▼

It follows that $\partial\sigma_x \in C_{n-1}(U_x - \{x\})$ for all $x \in f^{-1}(y)$.

We claim that $\sigma_x \in C_n(U_x, U_x - \{x\})$ is a generator of $H_n(U_x, U_x - x)$.

$$\begin{tikzcd}
H_n(X) \arrow[r, "\cong"] & H_n(X, X-x) \\
 & H_n(U_x, U_x - x) \arrow[u, "\text{excision}"']
\end{tikzcd}$$

$\sigma_x \in Z_n(U_x, U_x - x)$

$$f_*(\overline{\sigma_x}) = \deg_x(f) \cdot \theta_Y$$

where $\overline{\sigma_x}$ denotes the corresponding homology class.

$\square$

Corollary. *Let X and Y be as before, and now assume that f is C^∞. Suppose that $y \in Y$ is a regular value of f. Then $f^{-1}(y)$ is finite and $\deg_x(f) = \pm 1$ for all $x \in f^{-1}(y)$.*

Proof. For each $x \in f^{-1}(y)$, the restriction $(U_x, U_x - x) \to (U'_y, U'_y - y)$ is a diffeomorphism, which induces $H_n(U_x, U_x - x) \xrightarrow{\cong} H_n(U'_y, U'_y - y)$, so $\deg_x(f) = \pm 1$. $\square$

I haven't shown you how to determine whether the local degree is $+1$ or -1, but it is still useful to compute modulo 2. For f as above (we can drop the hypothesis that X and Y are orientable), then if y is a regular value of f and $\#f^{-1}(y) \notin 2\mathbb{Z}$, then f is not null-homotopic.

Example. Let $P \in \mathbb{C}[z]$ be a polynomial with $\deg(P) = d > 0$. Because the map $P : \mathbb{C} \to \mathbb{C}$ is proper, it extends to one-point-compactifications, so we have

$$\begin{tikzcd}
 & \mathbb{C} \arrow[r, "P"] \arrow[d, hook] & \mathbb{C} \arrow[d, hook] & \\
\mathbb{S}^2 \arrow[r, "\cong"] & \mathbb{C} \cup \{\infty\} \arrow[r, "P^*"'] & \mathbb{C} \cup \{\infty\} \arrow[r, "\cong"] & \mathbb{S}^2
\end{tikzcd}$$

and $\deg(P^*) = d$.

▼ Why did I write this? ▼

For all $a \in \mathbb{C}$, we have $\#P^{-1}(a) = d$. $P(z) - P(a)$, $P'(z) \neq 0$ (?)

Vector bundles

Definition. A map $p : E \to B$ is a fiber bundle if there is an open covering $\{U_\alpha\}_{\alpha \in A}$ of B, homeomorphisms $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ where F is a (possibly varying) topological space, such that for each $\alpha \in A$, the following diagram commutes

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "p|_{p^{-1}(U_\alpha)}"'] & & U_\alpha \times F \arrow[dl, "p_1"] \\
 & U_\alpha &
\end{tikzcd}$$

If B is connected, then all of the spaces F must be homeomorphic, and we say that p is a fiber bundle with fiber F.

Let U_α, U_β be two elements of this open cover of B. We can construct a commutative diagram:

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[dr] & p^{-1}(U_\alpha \cap U_\beta) \arrow[l, "\psi_\beta|_{p^{-1}(U_\alpha \cap U_\beta)}"'] \arrow[r, "\psi_\alpha|_{p^{-1}(U_\alpha \cap U_\beta)}"] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\
 & U_\alpha \cap U_\beta &
\end{tikzcd}$$

Let $\phi_{\alpha\beta} = \psi_\alpha|_{p^{-1}(U_\alpha \cap U_\beta)} \circ \psi_\beta|_{p^{-1}(U_\alpha \cap U_\beta)}^{-1}$, which is a homeomorphism making the following diagram commute:

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[rr, "\phi_{\alpha\beta}"] \arrow[dr] & & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\
 & U_\alpha \cap U_\beta &
\end{tikzcd}$$

When it is the case that that $\phi_{\alpha\beta}$ is a *diffeomorphism* for all α, β, we will say that "$U_\alpha \cap U_\beta \to \mathrm{Diffeo}(F)$".

Note that for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x).$$

This is called the cocycle condition.

Conversely, given an open cover $\{U_\alpha\}$, and maps $\phi_{\alpha\beta}$ satisfying the cocycle condition, we can construct a fiber bundle $E \to B$ by taking $\coprod_{\alpha \in A}(U_\alpha \times F)$, and quotienting by the equivalence relation $(x, v) \in U_\alpha \times F \sim (x, \phi_{\alpha\beta}(x)v) \in U_\beta \times F$ for all $x \in U_\alpha \cap U_\beta$ and $v \in F$.

Now we will define the notion of a vector bundle. We let $F = \mathbb{R}^k$ (or $\mathbb{C}^k$ for complex vector bundles), and we require that $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ is a diffeomorphism for all $\alpha, \beta \in A$, so that $(x, v) \mapsto (x, \phi_{\alpha\beta}(x)v)$ is a diffeomorphism from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself. The resulting fiber bundle on B is then a vector bundle. The number k is referred to as the rank of the vector bundle.

Given a C^∞ fiber bundle $p : V \to B$, then p is a vector bundle if

(a) there is an "addition" map

$$\begin{tikzcd}
V \times_B V \arrow[r] \arrow[dr] & V \arrow[d] \\
& B
\end{tikzcd}$$

(b) and there is a "scalar multiplication" map

$$\begin{tikzcd}
\mathbb{R} \times V \arrow[r] \arrow[dr] & V \arrow[d] \\
& B
\end{tikzcd}$$

that gives $p^{-1}(b)$ the structure of a vector space for all $b \in B$.

Basically any construction you can do with vector spaces, you can do with vector bundles: direct sum, tensor product, exterior algebra, etc.

Lecture 10 (2013-01-30)

To construct a vector bundle on a base B, we can take an open cover $B = \bigcup_{i \in I} U_i$ and appropriate transition functions $\varphi_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R})$ (appropriate just means that if we are doing topological manifolds, they should be continuous; C^∞ manifolds, they should be C^∞; etc.) that satisfy the cocycle condition $\varphi_{ij}(x)\varphi_{jk}(x) = \varphi_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$.

We then construct the vector bundle as

$$V := \coprod_{i \in I} (U_i \times \mathbb{R}^k) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by

$$(x, y) \in U_i \times \mathbb{R}^k \sim (x, \varphi_{ji}(x)) \in U_j \times \mathbb{R}^k \text{ for all } x \in U_i \cap U_j.$$

The bundle map $p : V \to B$ is the map induced by the projection maps $p_1 : U_i \times \mathbb{R}^k \to U_i$.

Given vector bundles $p_1 : V_1 \to B$ and $p_2 : V_2 \to B$, and letting $V_i(x) = p_i^{-1}(x)$ for each $x \in B$, we can construct new vector bundles

$$V_1 \oplus V_2, \quad V_1 \otimes_{\mathbb{R}} V_2, \quad V^*, \quad \Lambda^m(V), \quad \text{etc.}$$

whose fibers at $x \in B$ are $V_1(x) \oplus V_2(x)$, $V_1(x) \otimes_{\mathbb{R}} V_2(x)$, etc.

For example, we normally define $V_1 \oplus V_2 = V_1 \times_B V_2$, but here is another way: if V_1 and V_2 are the vector bundles constructed by $\varphi'_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R})$ and $\varphi''_{ij} : U_i \cap U_j \to \mathrm{GL}_\ell(\mathbb{R})$ respectively, we compose with the group homomorphism $\rho : \mathrm{GL}_k(\mathbb{R}) \times \mathrm{GL}_\ell(\mathbb{R}) \to \mathrm{GL}_{k+\ell}(\mathbb{R})$ sending

$$(A', A'') \longmapsto \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix},$$

and we see that $\rho \circ (\varphi'_{ij}, \varphi''_{ij})$ is a map from $U_i \cap U_j$ to $\mathrm{GL}_{k+\ell}(\mathbb{R})$ that also satisfies the cocycle condition. Then $V_1 \oplus V_2$ is the associated vector bundle to this information.

For the tensor product, the construction is essentially the same; we now take the homomorphism $\rho : \mathrm{GL}_k(\mathbb{R}) \times \mathrm{GL}_\ell(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{R}^\ell)$ sending (A', A'') to $A' \otimes A''$.

For V^*, we take $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}_k(\mathbb{R})$ defined by $\rho(A) = (A^{\mathsf{T}})^{-1}$.

For $\Lambda^m(V)$, we take $\rho : \mathrm{GL}(\mathbb{R}^k) \to \mathrm{GL}(\Lambda^m(V))$ defined by $\rho(T) = \Lambda^m(T)$.

Definition. We define the bundle of frames of a vector bundle as follows. Let $p : V \to B$ be a real vector bundle of rank k. We can construct the k-fold fiber product $V \times_B \cdots \times_B V$, and we define

$$P = \{(v_1, \ldots, v_k) \in V \times_B \cdots \times_B V \mid v_1, \ldots, v_k \text{ are linearly independent}\}.$$

This is an open subset of the k-fold fiber product. It consists of the smoothly consistent ways of taking isomorphisms $p^{-1}(x) \xleftarrow{\cong} \mathbb{R}^k$ for each x. Note that for $\psi \in P$ and $g \in \mathrm{GL}_k(\mathbb{R})$, we get $\psi g \in P$. Thus, we have a fiber bundle $\pi : P \to B$ with a right G-action on P, under which each fiber is stable (in fact the action on each fiber is simply transitive); in other words, the diagram

$$\begin{tikzcd} P \times G \arrow[r, "\text{action}"] \arrow[d, "p_1"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & B \end{tikzcd}$$

commutes. Such a fiber bundle is, by definition, a principal G-bundle on B. You can see some different takes on this in Kobayashi and Nomizu's *Foundations of Differential Geometry*, and Steenrod's *Topology of Fiber Bundles*.

Definition. Given a topological group G, a principal G-bundle P on B, and an action of G on a topological space F, the associated fiber space $P \times_G F$ is defined to be $(P \times F)/{\sim}$ where $\sim$ is the equivalence relation generated by

$$(zg, y) \sim (z, gy) \text{ for all } z \in P,\ y \in F, \text{ and } g \in G.$$

In fact, $P \times_G F$ is a fiber bundle on B with fiber F. The following diagram commutes:

$$\begin{tikzcd}
P \times F \arrow[d] \arrow[dr] & \\
P \arrow[d] & (P \times F)/{\sim} \arrow[dl] \\
B &
\end{tikzcd}$$

Given a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{R})$, we obtain an action of G on $\mathbb{R}^n$. Taking our fiber F to be $\mathbb{R}^n$, we can construct the associated fiber space $P \times_G \mathbb{R}^n$, which is a vector bundle on B. This is an equivalent way of constructing vector bundles - all of the constructions mentioned above can be done in this way too. For example, given a vector bundle V, we can construct $\mathrm{Sym}^m(V)$ by taking the associated fiber space to the natural homomorphism $\rho : \mathrm{GL}(\mathbb{R}^k) \to \mathrm{GL}(\mathrm{Sym}^m(\mathbb{R}^k))$.

Let X be a topological space that is decent enough to have a universal covering space $\pi : \widetilde{X} \to X$, and let the group of covering transformations Γ be given the discrete topology. Then $\widetilde{X}$ is a principal Γ-bundle on X, and we can consider associated fiber spaces, vector bundles, etc.

We often have an action of $\pi_1(X, x_0)$ on things. For example, if $p : E \to B$ is a fiber bundle and B is a manifold or simplicial complex, then $H^i(p^{-1}(x_0))$ is an abelian group with an action of $\pi_1(B, x_0)$. For any $\gamma \in \pi_1(X, x_0)$, the corresponding action arises from the commutative diagram

$$\begin{tikzcd}
\gamma^* E := I \times_B E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "p"] \\
I = [0,1] \arrow[r, "\gamma"'] & B
\end{tikzcd}$$

Here are the details of the construction.

Lemma. *Every fiber bundle on I is trivial.*

Proof. There is a finite open cover of I where the fiber bundle is trivial on each element (because I is compact). By induction on the number of elements on the cover, we are reduced to the case that there are two elements of the cover. WLOG, we'll assume that the fiber bundle is trivial on (open sets containing) $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, say with trivializations

$$\begin{tikzcd}
\pi^{-1}([0,\frac{1}{2}]) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & {[0,\frac{1}{2}] \times F} \arrow[dl, "p_1"] \\
& {[0,\frac{1}{2}]} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi^{-1}([\frac{1}{2},1]) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & {[\frac{1}{2},1] \times F} \arrow[dl, "p_1"] \\
& {[\frac{1}{2},1]} &
\end{tikzcd}$$

You can find a c such that

$$\begin{tikzcd}
 & F \\
\pi^{-1}(\frac{1}{2}) \arrow[ur, "\phi|", "\cong"'] \arrow[dr, "\psi|"', "\cong"] & \\
 & F \arrow[uu, "c"']
\end{tikzcd}$$

where $\phi|$ and $\psi|$ denote ϕ and ψ suitably restricted.

▼ Why is α smooth? ▼

There is then an isomorphism with the trivial bundle, $\alpha : E \to [0,1] \times F$, defined by

$$\alpha(e) = \begin{cases} \varphi(e) & \text{if } e \in \pi^{-1}([0, \frac{1}{2}]), \\ (\mathrm{id}_{[\frac{1}{2},1]} \times c)(\psi(e)) & \text{if } e \in \pi^{-1}([\frac{1}{2}, 1]). \end{cases} \qquad \square$$

▼ Are the isomorphisms independent of the trivialization of π chosen? ▼

Corollary 1. *If $\pi : E \to I$ is a fiber bundle, then the inclusions $\pi^{-1}(0) \to E$ and $\pi^{-1}(1) \to E$ are homotopy equivalences, so we get induced isomorphisms*

$$H^i(\pi^{-1}(0)) \xleftarrow{\cong} H^i(E) \xrightarrow{\cong} H^i(\pi^{-1}(1)).$$

▼ Aren't these homomorphisms in fact isomorphisms? ▼

Corollary 2. *Let $\pi : E \to B$ be a fiber bundle and let $\gamma : [0,1] \to B$ be a path from $\gamma(0) = x$ to $\gamma(1) = y$. We get an induced homomorphism $H^i(\pi^{-1}(x)) \to H^i(\pi^{-1}(y))$, as well as an induced homomorphism $H_i(\pi^{-1}(y)) \to H_i(\pi^{-1}(x))$.*

Proof. This is Corollary 1 applied to $\gamma^* E$. $\square$

We're out of time, so let me request that you read the definitions of direct limit, presheaf, stalk, sheaf, and sheafification. Spanier is a good reference for this.

Lecture 11 (2013-02-01)

Today we'll introduce a purely sheaf-theoretic definition of constructions such as the tensor product of bundles, which is the one that is used in practice.

Everything we talk about today will be C^∞.

Let $p : V \to B$ be a vector bundle on B. Given an open set $U \subseteq B$, a section s on U is a map $s : U \to V$ making the diagram commute:

$$\begin{tikzcd} & V \arrow[d, "p"] \\ U \arrow[ur, "s"] \arrow[r, hook] & B \end{tikzcd}$$

The set of C^∞ sections $s : U \to V$ is denoted by $\Gamma(U, V)$. We'll use the notation $C^\infty(V)$ for $\Gamma(B, V)$.

For any open set $U \subseteq B$, $\Gamma(U, V)$ is a module over the ring $C^\infty_B(U) = \{C^\infty \text{ maps } f : U \to \mathbb{R}\}$, and for any open sets $U' \subseteq U$ of B, the restriction map $\Gamma(U, V) \to \Gamma(U', V)$ is a homomorphism of $C^\infty_B(U)$-modules, where $\Gamma(U', V)$ is given the structure of a $C^\infty_B(U)$-module by restriction of scalars along the map $C^\infty_B(U) \to C^\infty_B(U')$. We say that $U \mapsto \Gamma(U, V)$ is a sheaf of modules over C^∞_B. Here is a diagram of the situation:

$$\begin{tikzcd} \Gamma(U,V) \arrow[d, "C^\infty(U) \text{ module hom.}"'] & \text{is a module over} & C^\infty(U) \arrow[d, "\text{ring hom.}"] \\ \Gamma(U',V) & \text{is a module over} & C^\infty(U') \end{tikzcd}$$

An example of a sheaf of C^∞_B-modules is

$$(C^\infty_B)^k = \underbrace{C^\infty_B \oplus \cdots \oplus C^\infty_B}_{k \text{ times}}.$$

If $p : V \to B$ is the trivial bundle of rank k, so that $V = B \times \mathbb{R}^k$ and p is the projection map, then clearly we can identify $C^\infty(V)$ with $(C^\infty_B)^k$.

Definition. A sheaf $\mathcal{F}$ of modules over C^∞_B is locally free of rank k if there is an open cover $B = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i} \cong (C^\infty_{U_i})^k$ for all $i \in I$.

Theorem. *The functor*

category of vector bundles on $B \to$ category of locally free, finite rank sheaves of C^∞_B-modules

which sends $V \mapsto C^\infty(V)$ is an equivalence of categories.

Proof. To see that it is fully faithful,

(I wasn't able to get down this part of the argument in class. The next page is a supplement provided to me by Professor Nori to fill in this gap.)

$C^\infty(V)$ denotes the sheaf of C^∞ sections of a vector bundle V on B. This is a sheaf of C^∞_B-modules on B. Given C^∞ vector bundles $p_i : V_i \to B$ for $i = 1, 2$ a homomorphism $f : V_1 \to V_2$ is a C^∞ map such that (i) $p_2 \circ f = p_1$ and (ii) for all $x \in B$ the map

$$f|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$$

is a linear transformation.

Given an open subset $U \subset B$ let $P(U)$ denote the set of homomorphisms from $(V_1)|_U$ to $(V_2)|_U$.

If U, U' are both open and if U' is a a subset of U then we have a restriction $P(U) \to P(U')$. This gives P the structure of a presheaf . The lemmas below are straightforward.

Lemma 0.1. *P is a sheaf*

Lemma 0.2. *Let F_1 and F_2 be sheaves on B. Let R be a sheaf of rings on B (we are concerned mainly with $R = C^\infty_B$). Let $Q(U)$ be the collection of $g : (F_1)|_U \to (F_2)|_U$ (here the g is required to be homomorphism of sheaves of $R|_U$-modules). The lemma states that presheaf Q on B is a sheaf.*

Lemma 0.3. *Let A be a ring. Every left A-module homomorphism $h : A \to A$ is of right multiplication by $h(1)$.*

Let R be a sheaf of rings on a space X. Every homomorphism h of sheaves of left R-modules from R to itself is right multiplication by $h(1) \in R(X)$.

Homomorphisms of sheaves of R-modules from R^k to R^l are given by $(k \times l)$ matrices with coefficients in the ring $R(X)$.

Definition 0.4. A C^∞ homomorphism $f : V_1 \to V_2$ gives rise to a homomorphism of sheaves of C^∞_B-modules $C^\infty(f) : C^\infty(V_1) \to C^\infty(V_2)$. Indeed if s is a C^∞ section of V_1 on an open subset U of B then $f \circ s$ is a section of V_2 defined on U. Thus $s \mapsto f \circ s$ gives rise to a homomorphism of presheaves, and that homomorphism is defined to be $C^\infty(f)$.

We put $F_i = C^\infty(V_i)$ and $R = C^\infty_B$ in the second lemma and obtain the sheaf Q on B.

Proposition 0.5. *With notation as above, $f \mapsto C^\infty(f)$ gives a bijection $P(B) \to Q(B)$.*

Proof. In fact the "$f \mapsto C^\infty(f)$ construction" is valid for every open subset U of B. In other words, we get a homomorphism of presheaves $P \to Q$. Let $\mathcal{U}$ be the collection of open subsets $U \subset B$ for which the vector bundles $(V_i)|_U$ are trivial bundles for both $i = 1, 2$.

Now the three statements
(a) $\mathcal{U}$ forms a basis for the topology of B
(b) both P and Q are sheaves
(c) $P(U) \to Q(U)$ is a bijection for all $U \in \mathcal{U}$ (this is the last assertion in lemma 3)

imply that $P(U) \to Q(U)$ is a bijection for all open subsets $U \subset B$, and in particular, for $U = B$.

□

We also need to check that this functor is essentially surjective. Let $\mathcal{F}$ be locally free. There exists an open cover of U_i's such that $\psi_i : \mathcal{F}_{U_i} \xrightarrow{\cong} (C^\infty_{U_i})^k$. Restricted to $U_i \cap U_j$, we get an isomorphism

$$(C^\infty_{U_i\cap U_j})^k \xrightarrow[\cong]{\psi_j\circ\psi_i^{-1}} (C^\infty_{U_i\cap U_j})^k$$

which as we saw before corresponds to a C^∞ map $\varphi_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R})$. Check that φ_{ij} satisfy the cocycle condition, which will then let us construct the required V. ☐

Let V_1, V_2 be vector bundles, and let $s_1 : U \to V_1$ and $s_2 : U \to V_2$ be C^∞ sections of V_1 and V_2 respectively on U. Then whatever the definitions are, we would like to be able to say that

"$s_1 \otimes s_2$ is a C^∞ section of $V_1 \otimes V_2$ on U".

Recall that we can take the tensor product of two sheaves. The sheaf $C^\infty(V_1) \otimes_{C^\infty_B} C^\infty(V_2)$ is a locally free sheaf of C^∞_B-modules. By the equivalence we just proved, we get a vector bundle Z with an isomorphism

$$C^\infty(Z) \xrightarrow[\cong]{T} C^\infty(V_1) \otimes C^\infty(V_2),$$

and we define Z to be $V_1 \otimes V_2$. We have that $s_1 \otimes s_2$ is a section of the bundle on the right on U, and $T^{-1}(s_1 \otimes s_2)$ is a section of Z on U.

Let V be a vector bundle on B of rank k. We define $\det(V) = \Lambda^k(V)$, the top exterior power of V, which is a line bundle on B (i.e., a vector bundle of rank 1).

From now on, assume that all our topological spaces Z to be connected, and satsify the following property $(*)$: the collection of contractible open subsets of Z forms a basis for the topology of Z.

Recall the definition of a constant presheaf: given an abelian group A, we define $P_A(U) = A$ for all U open on Y. The sheafification of P_A is denoted by A_Y.

Definition. The higher direct image is defined as follows. Given a continuous map $f : X \to Y$, fix an integer $q \geq 0$. For any open $U \subseteq Y$, we define $P^q(U) := H^q(f^{-1}(U))$. Of course, for $U' \subseteq U$, we have $f^{-1}(U') \subseteq f^{-1}(U)$, and cohomology is contravariant, which is the right direction. Thus, P^q is a presheaf on Y.

The qth higher direct image of f is the sheafification of P^q, and we denote it by $R^q f_* \mathbb{Z}_X$. As the sheafification, it satisfies a certain universal property: for any sheaf $\mathcal{G}$ on Y and $h : P^q \to \mathcal{G}$, there is a unique $\widetilde{h}$ such that

$$\begin{tikzcd}
P^q \arrow[r, "h"] \arrow[d] & \mathcal{G} \\
R^q f_* \mathbb{Z}_X \arrow[ur, dashed, "\widetilde{h}"'] &
\end{tikzcd}$$

What if $f : X \to Y$ is a fiber bunde with fiber F? In particular, we might be interested in the case when $X = Y \times F$ and $f = p_1$.

Because cohomology is contravariant, we get a map $H^q(F) \xrightarrow{p_2^*} H^q(Y \times F)$. By definition, we have $H^q(Y \times F) = P^q(Y)$, which comes with a restriction map to $P^q(U) = H^q(U \times F)$.

We claim that there is a natural map $H^q(F)_Y \xrightarrow{\phi} R^q f_* \mathbb{Z}_X$, and that this is an isomorphism.

The natural map ϕ is defined by

$$\begin{tikzcd}
P_{H^q(F)}(U) = H^q(F) \arrow[r] \arrow[dr] & P^q(U) \arrow[r] & (R^q f_* \mathbb{Z}_X)(U) \\
& H^q(F)_Y \arrow[ur, "\phi"'] &
\end{tikzcd}$$

To prove ϕ is an isomorphism of sheaves, it suffices to show it induces isomorphisms on all stalks.

Because the stalk at a point $y \in Y$ is the limit over all open sets containing y, and because Y satisfies the property $(*)$, we have

$$\begin{tikzcd}
H^q(F) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & R^q f_* \mathbb{Z}_Y \arrow[r, "\cong"] & H^q(F)
\end{tikzcd}$$

Corollary. *The sheaf $R^q f_* \mathbb{Z}_Y$ is locally constant, with stalks $\cong H^q(F)$.*

Definition. Let $p : V \to B$ be a real vector bundle of rank k. Let $0_V : B \to V$ be the zero section, and let $V' = V - 0_V(B)$. Then we can make a new bundle $p : (V, V') \to B$, with fibers $(\mathbb{R}^k, \mathbb{R}^k - \{0\})$. For any open set $U \subseteq B$, the map $U \mapsto \mathcal{P}(U) := H^k(p^{-1}(U), p^{-1}(U) \cap V')$ is a presheaf; take the associated sheaf. This sheaf is Or_V, the orientation sheaf of V.

Note that Or_V is a locally constant sheaf with all stalks $\cong \mathbb{Z}$. An orientation of V is a global section s of Or_V such that $s(b) \in H^k(p^{-1}(b), p^{-1}(b) - \{0\})$ is a generator for all $b \in B$.

Theorem (Thom isomorphism theorem)**.** *There is a natural isomorphism*

$$H^i(B, \mathrm{Or}_V) \xrightarrow{\cong} H^{i+k}(V, V')$$

where the left denotes cohomology with values in an abelian group (the sheaf Or_V *is locally constant, so this makes sense), and the right is cohomology of the pair (V, V').*

Theorem (Thom isomorphism theorem, for us)**.**

1. *$H^i(V, V') = 0$ for all $i < k$, when V is a vector bundle of rank k.*
2. *The natural map $H^k(V, V') \to \Gamma(B, \mathrm{Or}_V)$ is an isomorphism.*
3. *Let s be an orientation of V, i.e. $s \in \Gamma(B, \mathrm{Or}_V)$. Then s has a preimage $\theta \in H^k(V, V')$ by part 2. The map*

$$\begin{aligned} H^i(B) &\longrightarrow H^{i+k}(V, V') \\ \alpha &\longmapsto p^*\alpha \smile \theta \end{aligned}$$

   is an isomorphism.

Recall that we can form the cohomology ring $\bigoplus_{m=0}^{\infty} H^m(X)$, where cup product is the multiplication. For any $Y \subseteq X$, the group $\bigoplus_{m=0}^{\infty} H^m(X, Y)$ is a module over the cohomology ring.

Lecture 12 (2013-02-04)

Last time, we stated the Thom isomorphism theorem. Here are the first two parts:

Theorem. *Given a real vector bundle* $p : V \to B$ *of rank* k*,*

1. $H^i(V, V') = 0$ *for all* $i < k$*.*
2. *The natural map* $H^k(V, V') \to \Gamma(B, \mathrm{Or}_V)$ *is an isomorphism.*

Proof. We proceed by induction on k. Also, note that to prove the theorem for B, it will of course suffice to prove it for all open subsets of B; if we set

$$\mathcal{P}^i(U) = H^i(p^{-1}(U), p^{-1}(U) \cap V'),$$

then claim 1 is the same as saying that $\mathcal{P}^i(U) = 0$ for all $i < k$ and all U.

Given open subsets $U_1, U_2 \subseteq B$, we apply the Mayer-Vietoris sequence

$$\cdots \longrightarrow \mathcal{P}^{i-1}(U_1 \cap U_2) \longrightarrow \mathcal{P}^i(U_1 \cup U_2) \longrightarrow \mathcal{P}^i(U_1) \oplus \mathcal{P}^i(U_2) \longrightarrow \mathcal{P}^i(U_1 \cap U_2) \longrightarrow \cdots$$

Let $\mathcal{U} = \{\text{open } U \subseteq B \mid V|_U \text{ is a trivial vector bundle}\}$. Thus, for each $U \in \mathcal{U}$, we have some isomorphism of vector bundles

$$\begin{tikzcd}
V|_U = p^{-1}(U) \arrow[rr] \arrow[dr, "p|_{p^{-1}(U)}"'] & & U \times \mathbb{R}^k \arrow[dl, "p_1"] \\
& U &
\end{tikzcd}$$

and so $\mathcal{P}^i(U) \cong H^i(U \times (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}))$. Because $(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ has only one non-zero cohomology term and it is a free abelian group, we can apply the Künneth formula, and we see that this is then isomorphic to $H^{i-k}(U)$.

We prove claims 1 and 2 for each U of the form $U = U_1 \cup \cdots \cup U_m$, with $U_i \in \mathcal{U}$, by induction on m. The statement is true for $W = U_2 \cup \cdots \cup U_m$ by induction, and $U \cap W \subseteq U_1 \implies U \cap W \in \mathcal{U} \implies$ proved by induction hypothesis.

Thus, if claim 1 is true for $U_1 \cap B$, U_1, and W, then it is true for $U = U_1 \cup W$. The Mayer-Vietoris sequence for U_1 and W is

$$\cdots \longrightarrow \mathcal{P}^{i-1}(U_1 \cap W) \longrightarrow \mathcal{P}^i(U) \longrightarrow \mathcal{P}^i(U_1) \oplus \mathcal{P}^i(W) \longrightarrow \mathcal{P}^i(U_1 \cap W) \longrightarrow \cdots$$

Put $i = k$ for claim 2. We have isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}^k(U) \arrow[r] \arrow[d, "t"] & \mathcal{P}^k(U_1) \oplus \mathcal{P}^k(W) \arrow[r] \arrow[d, "\cong"] & \mathcal{P}^k(U_1 \cap W) \arrow[d, "\cong"] \\
& \Gamma(U, \mathrm{Or}_V) \arrow[r] & \Gamma(U_1, \mathrm{Or}_V) \oplus \Gamma(W, \mathrm{Or}_V) \arrow[r] & \Gamma(U_1 \cap W, \mathrm{Or}_V)
\end{tikzcd}$$

(the maps on the bottom have a minus sign in one coordinate, as in the Mayer-Vietoris sequence). A priori, the bottom row is just a complex, but because Or_V is a sheaf, in fact it is an exact sequence, and we can write

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}^k(U) \arrow[r] \arrow[d, "t"] & \mathcal{P}^k(U_1) \oplus \mathcal{P}^k(W) \arrow[r] \arrow[d, "\cong"] & \mathcal{P}^k(U_1 \cap W) \arrow[d, "\cong"] \\
0 \arrow[r] & \Gamma(U, \mathrm{Or}_V) \arrow[r] & \Gamma(U_1, \mathrm{Or}_V) \oplus \Gamma(W, \mathrm{Or}_V) \arrow[r] & \Gamma(U_1 \cap W, \mathrm{Or}_V)
\end{tikzcd}$$

By the Five Lemma, we have that the map t is an isomorphism.

To see how this implies the claim for all open sets, see Spanier. Certainly, we are done when B is compact. In Hurewicz and Wallman's *Dimension Theory*, there is a proof that if B is an n-manifold, or a subset of an n-dimensional simplicial complex, then $B = \bigcup_{i=1}^{n+1} U_i$ for some $U_i \in \mathcal{U}$. ☐

Proof of 3rd claim of Thom isomorphism theorem. By claim 2, there is an isomorphism $H^k(V, V') \cong \Gamma(B, \mathrm{Or}_V)$. For the third part of the theorem, we start with an orientation s, i.e. an $s \in \Gamma(B, \mathrm{Or}_V)$ such that $s(x)$ is a generator of the stalk of Or_V at x for all $x \in B$. There is a corresponding $\theta \in H^k(V, V')$, via this isomorphism.

The third claim is that the map $t_i(B) : H^i(B) \to H^{i+k}(V, V')$ defined by $\alpha \mapsto p^*\alpha \smile \theta$ is an isomorphism (note that $p^*\alpha \in H^i(V)$). As before, it will suffice to prove this for nice open sets U, i.e. our goal is now to prove that $t_i(U) : H^i(U) \to H^{i+k}(p^{-1}(U), p^{-1}(U) \cap V')$ is an isomorphism.

We have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{P}^{i+k}(U) \arrow[r] & \mathcal{P}^{i+k}(U_1) \oplus \mathcal{P}^{i+k}(V) \arrow[r] & \mathcal{P}^{i+k}(U_1 \cap V) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^i(U) \arrow[r] \arrow[u, "t_i(U)"] & H^i(U_1) \oplus H^i(V) \arrow[r] \arrow[u, "\cong"', "t_i(U_1)\oplus t_i(V)"] & H^i(U_1 \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

and the Five Lemma, with induction, implies that $t_i(U)$ is an isomorphism, ☐

How can we decide when a vector bundle is orientable?

Giving a global section of a sheaf is the same as giving a global section on each connected component.

First, let's talk about orientations of vector spaces. Given a real vector space V of dimension k, the 2 generators of $H^k(V, V')$ are in natural correspondence with the two connected components of $(\Lambda^k V)'$.

Fix a generator $\xi \in H^1(\mathbb{R}, \mathbb{R} \setminus \{0\})$, and more generally, let $\xi^{(n)} \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ be a generator.

Given a non-zero $\omega \in \Lambda^k V$, there is some basis $v_1, \ldots, v_k$ such that $\omega = v_1 \wedge \cdots \wedge v_k$.

Let $T : \mathbb{R}^k \to V$ be the map sending e_i to v_i. There is an induced map on cohomology, which is a also an isomorphism:

$$\mathbb{H}^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \xleftarrow[\cong]{T^*} H^k(V, V').$$

Then $(T^*)^{-1}(\xi^{(k)}) \in H^k(V, V')$ is a generator.

As an exercise, show that $\{\varphi \in \mathrm{GL}_k(\mathbb{R}) \mid \det(\varphi) > 0\}$ is connected. The maps $\varphi : (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \to (\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ are all homotopic to each other.

Now let V be a real rank k vector bundle on B, and assume B is connected. Take the top exterior power of V, which we'll write as $\det(V) := \Lambda^k(V)$. This is a vector bundle of rank 1. The fiber bundle

$$\begin{tikzcd}(\det(V))' \arrow[d] \\ B\end{tikzcd}$$

either has two connected components, and each connected component gives an orientation of V, or it has one connected component, in which case V is not orientable.

Let Y be a C^∞ manifold, and A a closed C^∞ submanifold of codimension k. By excision, there is an isomorphism

$$H^*(Y, Y-A) \xrightarrow{\cong} H^*(U, U-A)$$

for any open $U \supset A$. There is a normal bundle $(TA)^\perp$, i.e. considering $T_aA \subset T_aY$, we have $TY|_A = TA \oplus (TA)^\perp$. The tubular neighborhood theorem implies that, for A compact, there is a diffeomorphism $((TA)^\perp_\epsilon, 0_{(TA)^\perp}) \xrightarrow{\cong} (U, A)$, and $(TA)^\perp = N$ is a rank k bundle on A.

Therefore, we have an isomorphism $H^{i+k}(N_\epsilon, N'_\epsilon) \cong H^i(A)$ (if N has been given an orientation).

An orientation of N, the normal bundle of A in Y, gives rise to a $\theta \in H^k(Y, Y-A)$.

Lecture 13 (2013-02-06)

What we ended with last time was a discussion of what an orientation of a vector space was.

Let V be a rank k real vector space. There are two sets, each with two elements, and there is a natural bijection between them: the first is $\mathrm{Or}_V := \{\text{generators of } H^k(V, V')\}$, and the second is $\pi_0((\det(V)')$, where $\det(V) = \Lambda^k(V)$ is the top exterior power. Let the natural bijection be

$$\mathrm{Or}_V \xrightarrow[\text{bijection}]{N(V)} \pi_0((\det(V))')$$

Given vector spaces V_1 and V_2, we have a natural commutative diagram of bijections

$$\begin{tikzcd}
\mathrm{Or}_{V_1} \times \mathrm{Or}_{V_2} \arrow[r, "N(V_1)\times N(V_2)"] \arrow[d, "\text{cross product}"'] & \pi_0((\det(V_1)') \times \pi_0((\det(V_2))') \arrow[d, "\text{tensor product}"] \\
\mathrm{Or}_V \arrow[r, "N(V)"] & \pi_0((\det(V))')
\end{tikzcd}$$

where $V = V_1 \times V_2$.

Recall that there is a natural map

$$H^k(X, A) \otimes H^\ell(Y, B) \to H^{k+\ell}((X, A) \times (Y, B)) = H^{k+\ell}(X \times Y, A \times Y \cup X \times B).$$

Putting $(X, A) = (V_1, V_1')$ and $(Y, B) = (V_2, V_2')$, we have $(X, A) \times (Y, B) = (V, V')$.

Let $k = \dim(V_1)$ and $\ell = \dim(V_2)$. There is a natural isomorphism

$$\begin{tikzcd}
\det(V_1) \otimes \det(V_2) \arrow[r, "\cong"] & \det(V_1 \times V_2) \\
\omega \otimes \eta \arrow[r, mapsto] & \omega \wedge \eta
\end{tikzcd}$$

(note that the order matters). Similarly, we can map

$$\begin{tikzcd}
\det(V_1)' \times \det(V_2)' \arrow[r, "\cong"] & \det(V_1 \times V_2)' \\
(\omega, \eta) \arrow[r, mapsto] & \omega \wedge \eta
\end{tikzcd}$$

This gives a map

$$\pi_0((\det(V_1))') \times \pi_0((\det(V_2))') \to \pi_0((\det(V))').$$

Now let's define an orientation of a vector bundle.

Let $p : V \to B$ be a rank k real vector bundle, and define $V(x) = p^{-1}(x)$ for all $x \in B$. There is a quotient map $(\det(V(x)))' \to \pi_0((\det(V(x)))')$. Take the disjoint union over $x \in B$; we get a quotient map from the line bundle $\det(V)$, minus its zero section:

$$\begin{tikzcd}
(\det(V))' \arrow[r, "\alpha"] \arrow[dr] & Q \arrow[d, "\pi"] \\
& B
\end{tikzcd}$$

where

$$Q := \coprod_{x \in B} \pi_0((\det(V(x)))').$$

We give Q the quotient topology from α. Because a vector bundle is locally trivial, Q is locally a product with $\mathbb{Z}/2\mathbb{Z}$, so π is a covering space with all fibers being two-element sets. It is easy to see that an orientation of V is a continuous section $s : B \to Q$.

Part 2 of the Thom isomorphism theorem says that the natural map $H^k(V, V') \to \Gamma(B, \mathrm{Or}_V)$. An orientation according to our definition lives in $\Gamma(B, \mathrm{Or}_V)$, but an equivalent definition (now that the theorem has been proved) is that an orientation of V is an element $\theta \in H^k(V, V')$ such that for all $x \in B$, the image of θ under the map $H^k(V, V') \to H^k(V(x), V(x)')$ is a generator.

The Thom isomorphism theorem for $\mathbb{Z}/2\mathbb{Z}$ coefficients states that there is a $\theta \in H^k(V, V'; \mathbb{Z}/2\mathbb{Z})$ such that its image in $H^k(V(x), V(x)'; \mathbb{Z}/2\mathbb{Z})$ is non-zero for all $x \in B$, and part 2 in this case says that the map

$$\begin{tikzcd} H^i(B; \mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^{i+k}(V, V'; \mathbb{Z}/2\mathbb{Z}) \\ \alpha \arrow[r, mapsto] & p^*\alpha \smile \theta \end{tikzcd}$$

is an isomorphism for all $i \in \mathbb{Z}$.

Let's recall that the tubular neighborhood theorem. Given a closed C^∞ submanifold $A \subset Y$, and letting $i : A \hookrightarrow Y$ be the inclusion, then the map $TA \hookrightarrow i^*TY$ is given by $i'(a) : T_aA \to T_aY$.

Depending on whether or not you're an algebraist or a … something else, you will use the quotient, or the orthogonal complement. I will think of myself as an algebraist for today.

We define $N_1 = i^*TY/TA$, which is a vector bundle on A. The tubular neighborhood theorem says that there is a neighborhood U_1 of $0_{N_1}(A)$ and a neighborhood U_2 of A in Y such that there is a diffeomorphism $\varphi : U_1 \to U_2$ such that $\varphi(0_{N_1}(a)) = a$ for all $a \in A$.

Remark. Let P and Q be C^∞ manifolds, and let $\varphi : P \to Q$ be a C^∞ map. Suppose that $A \subset P$ is a closed set (need not be a submanifold) such that $\varphi|_A : A \to \varphi(A)$ is a homeomorphism, and $\varphi(A)$ is closed. Also, suppose that $\varphi'(a) : T_aP \to T_{\varphi(a)}Q$ is an isomorphism for all $a \in A$. Then there exist neighborhoods U_1 of A in O, and U_2 of $\varphi(A)$ in Q, such that $\varphi|_{U_1} : U_1 \to U_2$ is a diffeomorphism.

Proof. We will assume without proof the result that any second-countable, Hausdorff manifold Y can be realized as a closed submanifold of $\mathbb{R}^n$ for some n. Thus, we have

$$A \hookrightarrow Y \hookrightarrow \mathbb{R}^n,$$

and for any $a \in A$,

$$T_aA \subseteq T_aY \subseteq T_a\mathbb{R}^n = \mathbb{R}^n.$$

Then $N_1(a) = T_aY \cap (T_aA)^\perp$, and $N_2 = (T_aY)^\perp$ (note that all $\perp$'s are taken in $\mathbb{R}^n$). These are both C^∞ bundles on A, and $N_1 \times N_2 = N$ is the normal bundle of A in $\mathbb{R}^n$. (**When A is compact**) the old tubular neighborhood theorem states that there is some $\epsilon > 0$ such that there is a diffeomorphism $f : N_\epsilon \to U$, where U is an open subset of $\mathbb{R}^n$, $A \subset U$, and $f(0_N(a)) = a$ for all $a \in A$. We have

$$\begin{tikzcd}
p = U \cap Y \arrow[r, hook] & U \arrow[r, "f^{-1}"] & N_\epsilon \arrow[r, hook] & N \arrow[r, "\text{projection}"] & Q = N_1 \\
A \arrow[u, hook] \arrow[rrrr] & & & & 0_{N_1}(A) \arrow[u, hook]
\end{tikzcd}$$

On tangent space at all $a \in A$, check that the above is an isomorphism.

We have a decomposition

$$\begin{aligned} T_a U &= T_a A \oplus T_a(a + N_1(a) \cap U) \\ &= T_a A \oplus N_1(a) \end{aligned}$$ $\square$

Now, if Y is a C^∞ manifold and $A \subset Y$ is a closed C^∞ submanifold of codimension k, by excision we have

$$\begin{tikzcd}
H^k(Y, Y-A) \arrow[r, "\cong", "\text{excision}"'] & H^k(U_2, U_2 - A) \arrow[r, "\cong"] & H^k(U_1, U_1 - 0_{N_1}(A)) \\
 & & H^k(N_1, N_1') \arrow[u, "\cong", "\text{excision}"']
\end{tikzcd}$$

Given an orientation θ of the normal bundle of A in Y, by pushing it up and left, we get a canonical element (which we'll also call θ) of $H^k(Y, Y-A)$. We'd like to characterize this θ in a choice-free way. I claim that it has the property that for any locally closed submanifold Z of dimension k in Y such that $Z \cap A$ is a singleton and the intersection is transverse, the image of θ under

$$H^k(Y, Y-A) \to H^k(Z, Z-(Z \cap A))$$

is a generator.

The Thom-Gysin sequence is

$$\begin{tikzcd}
\cdots \arrow[r] & H^{m-1}(A) \arrow[r] & H^k(Y) \arrow[r] & H^m(Y-A) \arrow[r] & H^{m-k+1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Given C^∞ manifolds X and Y, a C^∞ map $f : X \to Y$, and a closed C^∞ submanifold $A \subset Y$ such that f is transverse to A, then we have a commutative diagram

$$\begin{tikzcd}
\theta(f^{-1}(A)) \in & H^k(X, X - f^{-1}(A)) \arrow[r] & H^k(X) \\
\theta(A) \in \arrow[u, mapsto] & H^k(Y, Y-A) \arrow[u] \arrow[r] & H^k(Y) \arrow[u]
\end{tikzcd}$$

The class $\theta(A)$ is denoted $\mathrm{cl}(A)$. We have $f^*(\mathrm{cl}(A)) = \mathrm{cl}(f^{-1}(A))$.

Lecture 14 (2013-02-08)

Finding invariants for the tangent and normal bundles will be important even for solving some classical problems. Today, we'll talk about some generalities on vector bundles.

Let $p : V \to B$ be an oriented real vector bundle of rank k. Recall that one way of defining an orientation, after having proved the Thom isomorphism, was a class $\theta \in H^k(V, V')$ such that its image under the map $H^k(V, V') \to H^k(V(x), V(x)')$ is a generator for all $x \in B$.

Because $p : V \to B$ is a homotopy equivalence (all fibers are Euclidean spaces which vary smoothly), the induced map on cohomology $p^* : H^k(B) \to H^k(V)$ is an isomorphism. A homotopy inverse for p is any section $s : B \to V$.

$$\begin{tikzcd}
\theta(V) \in H^k(V, V') \arrow[r, "M_V"] & H^k(V) \arrow[d, bend left, "s^*"] \\
& H^k(B) \arrow[u, bend left, "p^*"]
\end{tikzcd}$$

We define the Euler class of $p : V \to B$ to be

$$\begin{aligned} e(V) :=& (p^*)^{-1} M_V \theta(V) \\ =& s^* M_V \theta(V) \end{aligned}$$

Thus, $e(V)$ is an element of $H^k(B)$.

Proposition.

1. *If V_1 and V_2 are oriented vector bundles on B that are isomorphic to each other (as oriented bundles), then $e(V_1) = e(V_2)$.*
2. *(Functoriality) Given $h : C \to B$ and a vector bundle $p : V \to B$, an orientation $\theta(V)$ of V induces an orientation of h^*V, the pullback bundle:*

$$\begin{tikzcd}
h^*V \arrow[r, "\widetilde{h}"] \arrow[d] & V \arrow[d, "p"] \\
C \arrow[r, "h"'] & B
\end{tikzcd}$$

   and $h^*e(V) = e(h^*V)$.

3. *(Direct sums) If V_1 and V_2 are oriented vector bundles over B, then $V_1 \oplus V_2$ is as well, and $e(V_1) \smile e(V_2) = e(V_1 \oplus V_2)$. More generally, if V_1 is an oriented vector bundle on B_1 and V_2 is an oriented vector bundle on B_2, then we can construct the product bundle*

$$\begin{tikzcd}
V = V_1 \times V_2 \arrow[d, "p = p_1 \times p_2"] \\
B = B_1 \times B_2
\end{tikzcd}$$

   Note that $(V_1, V_1') \times (V_2, V_2') = (V, V')$, and define $\theta(V) = \theta(V_1) \times \theta(V_2)$, where $\theta(V_i) \in H^{k_i}(V_i, V_i')$ for $i = 1, 2$. We claim that $e(V_1 \times V_2) = e(V_1) \times e(V_2) \in H^{k_1+k_2}(B_1 \times B_2)$.

Exercise: Prove that $e(V) = (-1)^k e(V)$.

Proof. We have a commutative diagram for any $x \in B$ and $y \in C$ with $h(y) = x$:

$$\begin{tikzcd}
\theta(V) \in \arrow[d, mapsto] & H^k(V, V') \arrow[r, "h^*"] \arrow[d] & H^k(h^*V, (h^*V)') \arrow[d] \\
\text{generator} \in & H^k(V(h(x)), V(h(x))') \arrow[r, "\cong"'] & H^k((h^*V)(y), (h^*V)(y)')
\end{tikzcd}$$

Thus, $h^*\theta(V)$ is an orientation of h^*V, proving claim 1. Observe that a section $s : B \to V$ induces a section of the pullback:

$$\begin{tikzcd}
h^*V \arrow[r] \arrow[d] & V \arrow[d, "p"'] \\
c \arrow[r, "h"'] \arrow[u, bend right, "h^*s"'] & B \arrow[u, bend right, "s"']
\end{tikzcd}$$

To prove claim 2 (functoriality), consider the following diagram:

$$\begin{tikzcd}
H^k(V, V') \arrow[r, "\widetilde{h}^*"] \arrow[d, "M_V"'] & H^k(h^*V, (h^*V)') \arrow[d, "M_{h^*V}"] \\
H^k(V) \arrow[r] \arrow[d, "s^*"'] & H^k(h^*V) \arrow[d, "(h^*s)^*"] \\
H^k(B) \arrow[r, "h^*"'] & H^k(C)
\end{tikzcd}$$

Proof of claim 3: By the functoriality of cross product, we are done - we have the following commutative diagram:

$$\begin{tikzcd}
B_1 \arrow[d, "s_1"] & \times & B_2 \arrow[r] \arrow[d, "s_2"] & B_1 \times B_2 \arrow[d, "s_1 \times s_2"] \\
V_1 \arrow[d] & \times & V_2 \arrow[r] \arrow[d] & V \arrow[d] \\
(V_1, V_1') & \times & (V_2, V_2') \arrow[r] & (V, V')
\end{tikzcd}$$

For $V_1 \oplus V_2$, we have $B_1 = B_2$ above, and

$$\begin{tikzcd}
V_1 \oplus V_2 = \Delta B^*(V_1 \times V_2) \arrow[r] \arrow[d] & V_1 \times V_2 \arrow[d] \\
B \arrow[r, "\Delta B"'] & B \times B
\end{tikzcd}$$

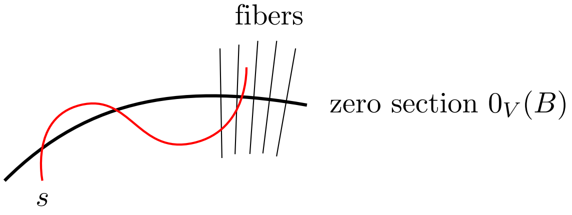
and

$$\begin{aligned}
e(V_1 \oplus V_2) &= \Delta B^*(e(V_1 \times V_2)) \\
&= (\Delta B)^*(e(V_1) \times e(V_2)) \\
&= e(V_1) \smile e(V_2)
\end{aligned}$$

□

Interpretation for C^∞ bundles V on C^∞ manifolds B

Recall that we stated without proof that there exists a C^∞ section s of V that is transverse to the zero section.



Define $Z(s) = \{x \in B \mid s(x) = 0\}$, which is a C^∞ submanifold. For any $x \in Z(s)$, there is a canonical identification

$$T_xB/T_xZ(s) \xrightarrow{\cong} T_xV/T_x0_V(B) \cong V(x).$$

We have

$$\begin{tikzcd}
\theta(V) \in H^k(V,V') \arrow[r, "M_V"] \arrow[dr, "s^*"'] & H^k(V) \arrow[r, "s^*"] & H^k(B) \ni e(V) \\
& H^k(B, B - Z(s)) \arrow[ur] & \\
& \theta(Z(s)) \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

Thus $e(V) = \mathrm{cl}(Z(s))$ for any section s which meets the zero section $0_V(B)$ transversely.

For example, if $B = \mathbb{S}^k$ for k odd, and $V = TB$ is the tangent bundle, then we have a section

$$s(x_1, \dots, x_{k+1}) = (x_2, -x_1, x_4, -x_3, \dots)$$

This is nowhere-vanishing, so we have that $e(T\mathbb{S}^k) = 0$ for k odd. Similarly, for k even, we have $e(T\mathbb{S}^k) = 2$.

Theorem (Hopf). *If X is a compact manifold, then $e(TX) =$ the Euler characteristic of X.*

Steenrod's *Topology of Fiber Bundles* section on obstruction theory is a good reference for this.

Theorem. *If B is a CW complex, with successive skeleta*

$$B^{(0)} \subset B^{(1)} \subset \cdots$$

and V is an oriented real vector bundle on B of rank k, then $e(V) = 0 \in H^k(B)$ if and only if there is a section of $V'|_{B^{(k)}}$,

$$\begin{tikzcd}
& V \arrow[d] \\
B^{(k)} \arrow[r, hook] \arrow[ur, "s"] & B
\end{tikzcd}$$

Complex Vector Bundles

The main difference between complex and real vector spaces is that complex vector spaces have a canonical orientation.

Exercise: Let V be a complex vector space of dimension k. If $v_1, \dots, v_k$ is a $\mathbb{C}$-basis for V, then we can define a map $\det_{\mathbb{C}}(V) \to \det_{\mathbb{R}}(V)$, i.e. a map $\Lambda^k_{\mathbb{C}}(V) \to \Lambda^{2k}_{\mathbb{R}}(V)$, by

$$S_v(v_1 \wedge \cdots \wedge v_k) = v_1 \wedge iv_1 \wedge \cdots \wedge v_k \wedge iv_k.$$

To check that this is well-defined, show that if we map $T \in \mathrm{M}_k(\mathbb{C})$ to the corresponding element of $\mathrm{M}_{2k}(\mathbb{R})$, then $|\det_{\mathbb{C}}(T)|^2 = \det_{\mathbb{R}}(T)$.

The canonical orientation on V is determined by the connected component which contains the image of the map

$$\begin{tikzcd}
\Lambda^k_{\mathbb{C}}(V)' \arrow[r] & \Lambda^{2k}_{\mathbb{R}}(V)' \\
\mathbb{C} \setminus \{0\} \arrow[r] & \mathbb{R} \setminus \{0\}
\end{tikzcd}$$

If V_1 and V_2 are complex vector spaces, then $S_{V_1} \times S_{V_2} \to S_{V_1 \oplus V_2}$, considering $S_V \in \pi_0(\det_{\mathbb{R}}(V)')$.

Theory of Chern Classes

For every complex vector bundle V on a topological space B, we define a Chern class $c_i(V) \in H^{2i}(B)$ for $i = 0, 1, \dots, \mathrm{rank}_{\mathbb{C}}(V)$, where $c_0(V) = 1 \in H^0(B)$, and then define the total Chern class to be

$$c(V) = c_0(V) + \cdots + c_k(V).$$

There are several reasonable properties we should check:

Proposition.

1. *If $V_1 \cong V_2$, then $c(V_1) = c(V_2)$.*
2. *If $p : V \to B$ is a complex vector bundle and $f : C \to B$ is smooth, then $f^*(c_i(V)) = c_i(f^*V)$.*
3. *(Whitney sum formula) We have $c(V_1 \oplus V_2) = c(V_1)c(V_2)$ (the product is the cup product), and specifically,*

   $$c_m(V_1 \oplus V_2) = c_m(V_1)c_0(V_2) + c_{m-1}(V_1)c_1(V_2) + \cdots + c_0(V_1)c_m(V_2).$$

4. *Let $\mathscr{L}$ be the tautological line bundle on $\mathbb{P}^n(\mathbb{C})$. Then $c_1(\mathscr{L}^*) = t$, where $t \in H^2(\mathbb{P}^n(\mathbb{C}))$ is the canonical generator, i.e. $t = \mathrm{cl}(\mathbb{P}^{n-1}(\mathbb{C}))$.*

The tautological line bundle $\mathscr{L}$ on $\mathbb{P}^n(\mathbb{C})$ is defined to be

$$\mathscr{L} = \{(L, v) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1} \mid v \in L\}.$$

Then $\mathscr{L}^*$ denotes the ($\mathbb{C}$-)dual bundle.

Theorem (Existence and uniqueness of the theory of Chern classes)**.**

Facts:

1. $c_1(V) = c_1(\det_{\mathbb{C}}(V))$
2. $\{$*isomorphism classes of complex line bundles on* $B\} \xrightarrow[\cong]{c_1} H^2(B;\mathbb{Z})$.
3. *If* V *is a complex vector bundle of rank* k*, then* $e(V) = c_k(V) \in H^{2k}(B;\mathbb{Z})$.

Proof of existence. Given a complex vector bundle V of rank k on B, then we can form

$$V \boxtimes \mathscr{L}^* = p_1^* V \otimes_{\mathbb{C}} p_2^* \mathscr{L}^*,$$

which is a complex vector bundle on $B \times \mathbb{P}^n(\mathbb{C})$. We have

$$e(V \boxtimes \mathscr{L}^*) = p_1^* c_k(V) p_2^* 1 + p_1^* c_{k-1}(V) p_2^* t + \cdots + p_1^* c_{k-i}(V) p_2^* t^i + \cdots \in H^{2k}(B \times \mathbb{P}^n(\mathbb{C});\mathbb{Z}),$$

where we have used the Kunneth formula to make the identification

$$H^m(B \times \mathbb{P}^n(\mathbb{C});\mathbb{Z}) \cong \bigoplus_{p+q=m} H^p(B) \otimes H^q(\mathbb{P}^n(\mathbb{C})) = H^m(B) \oplus H^{m-2}(B) \oplus \cdots.$$

In this identification, we're claiming that

$$e(V \boxtimes \mathscr{L}^*) \mapsto (c_k(V), c_{k-1}(V), \ldots).$$

For part 3, when $V = V_1 \oplus V_2$, we have

$$V \boxtimes \mathscr{L}^* = V_1 \boxtimes \mathscr{L}^* \oplus V_2 \boxtimes \mathscr{L}^*,$$

and

$$e(V \boxtimes \mathscr{L}^*) = e(V_1 \boxtimes \mathscr{L}^*) \smile e(V_2 \boxtimes \mathscr{L}^*).$$

□

Given a C^∞ map $X \to Y$, and $a \in Y$ a regular value of f, let $M = f^{-1}(a)$. For every $x \in M$, we have an identification

$$T_x X / T_x M \xrightarrow[\cong]{\overline{f'(x)}} T_x Y.$$

$c(\mathbb{C} \otimes_{\mathbb{R}} V$ is the Pontryagin class of V.

The normal bundle i^*TX/TM is trivial, so $i^*TX = \text{trivial bundle} \oplus TM$, and

$$c(\mathbb{C} \otimes i^*TM) = 1 \cdot c(\mathbb{C} \otimes TM).$$

If, for example, $X = \mathbb{R}^n$, then we would have

$$TM \oplus \text{trivial bundle} = \text{some other trivial bundle},$$

which doesn't imply that TM is trivial, but this is the definition of *stably* trivial. Trivial bundles will never affect the Chern class, so

$$c(\mathbb{C} \otimes TM) = 1, \text{ i.e. } c_i(\mathbb{C} \otimes TM) = 0 \text{ for all } i > 0.$$

The Chern class of the tangent bundle of $\mathbb{P}^n(\mathbb{C})$ is

$$c(T\mathbb{P}^n(\mathbb{C})) = (1+t)^{n+1},$$

and for $n \geq 2$, we have $c_2 = -(n+1)t^2$.

$$c(\mathbb{C} \otimes_{\mathbb{R}} T\mathbb{P}^n(\mathbb{C})) = (1+t)^{n+1}(1-t)^{n+1} = (1-t^2)^{n+1}.$$

Thus, for $n \geq 2$, the tangent bundle of $\mathbb{P}^n(\mathbb{C})$ cannot be the pullback of the tangent bundle of a manifold.

Lecture 15 (2013-02-11)

I'll be following Warner's book from now on, so I won't have to type up notes and so you'll have something to refer to. We'll be talking about flows and vector fields soon.

Continuous Variants of C^∞ Theorems

Theorem 1. *Let X and Y be C^∞ manifolds, and assume that X is compact. Let $f_0 : X \to Y$ be continuous. Then f_0 can be approximated by C^∞ functions; you could either use a metric on Y, or use the compact-open topology. Either way, there are C^∞ functions $f_n : X \to Y$ that converge to f_0.*

Proof. We will assume the Stone-Weierstrass theorem. Choose an embedding $X \hookrightarrow \mathbb{R}^n$. Because X is compact, the Stone-Weierstrass theorem says that $C^\infty(X)$ is dense in $C(X, \mathbb{R})$ in the $\|\cdot\|_\infty$ norm. Thus, the statement is true if $Y = \mathbb{R}$. It will then follow for $Y = \mathbb{R}^n$, then for $Y =$ an open subset of $\mathbb{R}^n$, and then for any $Y =$ any C^∞ retract of an open subset of $\mathbb{R}^n$. But from the tubular neighborhood theorem, this last case includes any Y; there is an embedding $i : Y \hookrightarrow \mathbb{R}^n$ and a retract $r : U \to Y$ for some open $U \supset i(Y)$. □

Theorem (Partitions of unity)**.** *Let M be a C^∞ manifold, and let $\mathcal{U}$ be an open cover of M. Then for any $U \in \mathcal{U}$, there is a C^∞ function $\varphi_U : M \to \mathbb{R}$ such that*

- *(a)* $\operatorname{supp}(\varphi_U) \subset U$
- *(b)* $\varphi_U \geq 0$

and such that the collection $\{\varphi_U \mid U \in \mathcal{U}\}$ satisfies the condition

$$\textit{for all } x \in M, \textit{ there is some neighborhood } U(x) \textit{ of } x \textit{ such that } \{U \in \mathcal{U} \mid \varphi_U|_{U(x)} \neq 0\} \textit{ is finite}$$

and $\sum_{U \in \mathcal{U}} \varphi_U(x) = 1$ for all $x \in M$.

Theorem 2. *Let X, Y, and f_0 be as in Theorem 1. The set*

$$U := \{x \in X \mid f_0|_{\textit{some nbhd of } x} \textit{ is } C^\infty\}$$

is open in X. Let $K \subset U$ be a closed subset. Then the sequence of f_n's from Theorem 1 can be chosen so that $f_n|_K = f_0|_K$.

Proof. We have a C^∞ map φ with $\operatorname{supp}(\varphi) \subset U$, and a C^∞ map ψ with $\operatorname{supp}(\psi) \subset X - K$, so that $\psi(x) = 0$ for all $x \in K$, and hence $\varphi(x) = 1$ for all $x \in V$, where V is a neighborhood of K.

When $Y = \mathbb{R}^n$, we have the f_n's as in Theorem 1: the f_n's are C^∞ and converge to f_0.

Let $g_n = \varphi f_0 + \psi f_n$ for all $n \in \mathbb{N}$. Then $\varphi f_0 \to f_0$ on U, f_0 is C^∞ on U, and ψf_n is a C^∞ map, so g_n is C^∞ on X and $g_n(x) = f_0(x)$ for all $x \in K$. Then we deal with open subsets of $\mathbb{R}^n$, C^∞ retracts of open subsets, etc.

□

Theorem 3. *Let X, Y, and f_0 be as in Theorem 1. Let Ω be a tubular neighborhood of Y, i.e.*

$$\begin{tikzcd} Y \arrow[r, hook, bend left, "i"] & \Omega \arrow[l, bend left, "r"] \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

Let $A \hookrightarrow Y$ be a closed C^∞ submanifold. Then the set

$$U = \{x \in X \mid \textit{there is some neighborhood } U(x) \textit{ such that } f_0|_{U(x)} \textit{ is transverse to } A\}$$

is open. Let $K \subset U$ be closed. Then there is a sequence $f_n : X \to Y$ of functions such that

- *(i) $f_n \to f_0$ in the compact-open topology*
- *(ii) the f_n's are C^∞ and transverse to A*
- *(iii) $f_n|_K = f_0|_K$ for all $n \in \mathbb{N}$*

Remark. By Theorem 2, we have C^∞ maps $f_n : X \to Y$ such that $f_n \to f_0$ uniformly and $f_n|_K = f_0|_K$. Thus, it suffices to prove that the f_n's can be approximated by f_n' which are C^∞, transverse to A, and $f_n'|_K = f_n|_K$. Thus, it suffices to prove Theorem 3 when f_0 is C^∞.

Proof. Recall our proof that C^∞ functions can be approximated by functions transverse to a given submanifold; we will modify this proof. We had our tubular neighborhood

$$\begin{tikzcd} Y \arrow[r, hook, bend left, "i"] & \Omega \arrow[l, bend left, "r"] \arrow[r, hook] & \mathbb{R}^n \end{tikzcd}$$

and considered the map $X \times S \to \mathbb{R}^n$ defined by $F(x,s) = f_0(x) + s$, where $S = \mathbb{R}^n$. But then we restrict S to be a neighborhood of 0 so that the image of F is entirely contained in Ω, and then redefine $F(x,s) = r(f_0(x) + s)$.

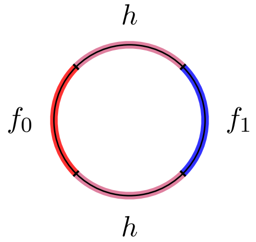
We have a partition of unity (φ, ψ) for the open cover $(U, X - K)$. Now we define $F(x,s)$ as

$$F(x,s) = r(\varphi(x) f_0(x) + \psi(x) s)$$

for $x \in X$, $s \in$ a neighborhood of 0 in $\mathbb{R}^n$. Let $f_s(x) = F(x,s)$. Then we have $V = \{x \in X \mid \psi(x) \neq 0\}$ and clearly $F'(x,0)$ is surjective for all $x \in V$, so it will be transverse to absolutely anything, and $f_s(x) = f_0(x)$ for all $x \in X - V$, and we assumed that f_0 was transverse to A. Thus, we can conclude that $F : X \times S \to Y$ is transverse to A on $X \times \{0\}$. Because X is compact, shrinking the neighborhood S if necessary we can assume that $F : X \times S \to Y$ is transverse to A everywhere. Thus $F^{-1}(A)$ is a manifold, we have $F^{-1}(A) \hookrightarrow X \times S \to S$, and now apply Sard's theorem. ☐

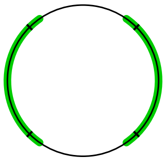
Given an element $\xi \in [X,Y]$ (this denotes the set of homotopy classes of continuous maps from X to Y) and a closed C^∞ submanifold $A \subset Y$, pick an $f \in \xi$ that is C^∞ and transverse to A. Suppose that $f_0, f_1 : X \to Y$ are both C^∞ and transverse to A, and that they are homotopic to each other via some h.

Define $F : X \times \mathbb{S}^1 \to Y$ by



Assume that $\dim(X) > 0$.

Let $U = X \times U'$ where U' is a neighborhood of the regions where F is either f_0 or f_1, as follows:



and let $K = X \times \{\pm 1\}$. Apply Theorem 3 to $F : X \times \mathbb{S}^1 \to Y$, get $\widetilde{F} : X \times \mathbb{S}^1 \to Y$ which is C^∞, transverse to A, and for all $x \in X$, we have

$$\widetilde{F}(x, -1) = f_0(x), \quad \widetilde{F}(x, 1) = f_1(x).$$

Then $\widetilde{F}^{-1}(A) = B$ is a C^∞ submanifold of $X \times \mathbb{S}^1$. We have $B \mathrel{\overline{\pitchfork}} X \times \{1\} = f_1^{-1}(A) \times \{1\}$ and $B \mathrel{\overline{\pitchfork}} X \times \{-1\} = f_0^{-1}(A) \times \{-1\}$.

The symbol $\overline{\pitchfork}$ means "is transverse to".

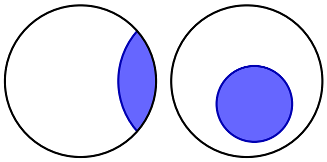
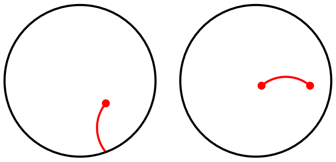
However, this use of $\mathbb{S}^1$ was really an artificiality on my part. We really should have said everything for manifolds with boundary, which I'll define now.

Consider the closed lower half plane

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\},$$

with its sheaf C^∞_H. An n-manifold with boundary is a topological space X such that for all $x \in X$ there is a neighborhood $U(x)$ of x in X and a neighborhood $U(h)$ of some $h \in H$, and a homeomorphism $U(x) \to U(h)$, and these homeomorphisms are compatible.

A closed subset A of a C^∞ manifold with boundary $(M, \partial M)$ if, for all $a \in A$, either $a \notin \partial M$ (this is just the old definition), or if $a \in A \cap \partial M$, then we require that $(M, \partial M)$ has a chart to H, $(x_1, \dots, x_n)$ with $x_n \leq 0$, such that in the chart, A is the set where $x_1 = \cdots = x_k = 0$ for some $k \neq n$, and such that $A \cap \partial M = \partial A$.

Submanifold with boundary?	Examples
Yes	
No	

Closed submanifolds B_0 and B_1 of a C^∞ manifold X are concordant if there is a closed submanifold B with boundary of $X \times I$ such that $B \cap X \times \{0\} = B_0 \times \{0\}$ and $B \cap X \times \{1\} = B_1 \times \{1\}$. This is an equivalence relation; let $\mathcal{C}$ denote the set of equivalence classes. The content of what we've proved is that there is a well-defined map

$$[X, Y] \times \mathcal{C}(Y) \to \mathcal{C}(X)$$

taking closed submanifolds in Y to their preimage in X.

Lecture 16 (2013-02-13)

Theorem (Ehresmann fibration theorem)**.** *Let $f : X \to Y$ be a C^∞ proper submersion. Then f is a C^∞ fiber bundle.*

Proof. Let $y_0 \in Y$, and let $F = f^{-1}(y_0)$, which is compact because f is proper. We may regard X as a closed submanifold of $\mathbb{R}^k$ for some k. Thus, we have $F \subset X \subset \mathbb{R}^k$, and we apply the tubular neighborhood theorem to $i : F \hookrightarrow \mathbb{R}^k$. Thus, we get an open neighborhood U of F in $\mathbb{R}^k$ and a C^∞ retraction $r : U \to F$. Now replace (U, r) by $(U \cap X, r|_{U\cap X})$.

We now have that U is a neighborhood of F in X, and $r : U \to F$ is still C^∞. Consider the map $(f|_U \times r) : U \to Y \times F$. We claim that this induces an isomorphism on tangent space at all $x \in F \subset U$. We see that the sequence

$$0 \longrightarrow T_x(f^{-1}(f(x))) \longrightarrow T_x(U) \xrightarrow{f'(x)} T_{f(x)}(Y) \longrightarrow 0$$

is exact for all $x \in X$ because f is a submersion, and in particular, for $x \in F = f^{-1}(y_0)$, we have that the sequence

$$0 \longrightarrow T_xF \longrightarrow T_xU \longrightarrow T_{y_0}Y \longrightarrow 0$$

is exact. Thus, the claim is now equivalent to the claim that

$$T_xF \xrightarrow{r'(x)|_{T_xF}} T_xF$$

is an isomorphism. Why? If we have a map of vector spaces

$$V \xrightarrow{(\alpha,\beta)} A \oplus B$$

and we know that α is onto, then (α, β) is an isomorphism if and only if $\beta|_{\ker(\alpha)} : \ker(\alpha) \to B$ is an isomorphism. But $r'(x)|_{T_xF}$ is the identity, so this is true in our case.

Note that the set

$$V = \{z \in U \mid f|_U \times r \text{ is an isomorphism of tangent spaces } T_zX \to (\cdots)\}$$

is open, so $X \setminus V$ is closed. Thus $f(X \setminus V)$ is closed, because f is proper. Its complement necessarily contains $f^{-1}(U_{y_0})$, where U_{y_0} is a neighborhood of y_0 in Y. In other words,

$$f^{-1}(U_{y_0}) \xrightarrow{f|\times r} U_{y_0} \times F$$

induces an isomorphism of tangent space and is injective on $F \subset f^{-1}(U_{y_0})$. It follows that $f \times r$ is injective on $f^{-1}(U'_{y_0})$ for some neighborhood $U'_{y_0} \subset U_{y_0}$ of y_0. □

Here is a variant of the theorem.

Theorem. *Let $(X, \partial X)$ be a C^∞ manifold with boundary. Suppose that $f : X \to Y$ is a C^∞ proper submersion such that $f|_{\partial B}$ is also a submersion. Then f is a C^∞ fiber bundle.*

Existence and uniqueness of solutions of ODEs

A reference for this is Warner's *Foundations of Differentiable Manifolds and Lie Groups.*

Definition. Let M be a C^∞ manifold. We say that A is a continuous vector field on M when A is a continuous section $A : M \to TM$ the tangent bundle map $p : TM \to M$. A C^∞ vector field is just a C^∞ section.

Definition. An integral curve of a vector field v on M is a C^1 curve $\gamma : (a, b) \to M$ such that $\gamma'(t) = v(\gamma(t))$ for all $t \in (a, b)$.

Theorem (Existence and uniqueness)**.** *Let v be a C^1 vector field on M. Let $x_0 \in M$. Then there exists an integral curve $\gamma : U \to M$ for some connected open neighborhood $U \subseteq \mathbb{R}$ of 0 such that $\gamma(0) = x_0$. If $\gamma_i : U_i \to M$ for $i = 1, 2$ are integral curves of v such that $\gamma_1(0) = \gamma_2(0) = x_0$, then there is some open neighborhood $U \subset U_1 \cap U_2$ of 0 such that $\gamma_1|_U = \gamma_2|_U$.*

Corollary. *In fact, $\gamma_1|_{U_1\cap U_2} = \gamma_2|_{U_1\cap U_2}$.*

Proof of corollary. Both U_1 and U_2 are connected open neighborhoods of 0 so it suffices to check the claim on the positive side and on the negative side. Let $U_i \cap [0, \infty) = [0, b_i)$ for $i = 1, 2$. We may assume WLOG that $b_1 \leq b_2$. Let

$$c = \sup\{t \geq 0 \mid \gamma_1(t) = \gamma_2(t)\}.$$

If $c = b_1$, then we are done. If $c < b_1$, then continuity implies that $\gamma_1(c) = \gamma_2(c)$. Consider the curves $\gamma_1(t + c)$ and $\gamma_2(t + c)$, which are both integral curves for v with the same initial value, i.e. they agree when $t = 0$. But the theorem implies that, for any integral curve δ of v with initial value $\gamma_1(c) = \gamma_2(c)$, we have that $\delta(t) = \gamma_1(t + c)$ for all t in a neighborhood of 0, and $\delta(t) = \gamma_2(t + c)$ for all t in a neighborhood of 0, and therefore $\gamma_1(t) = \gamma_2(t)$ for all $t \in (c - \epsilon, c + \epsilon)$ for some $\epsilon > 0$. But this contradicts the construction of c; thus $c = b_1$ and $\gamma_1|_{U_1\cap U_2\cap[0,\infty)} = \gamma_2|_{U_1\cap U_2\cap[0,\infty)}$. Now do the same for $(-\infty, 0]$.

Now consider the set of all pairs

$$Z = \left\{ (U, \gamma) \;\middle|\; \begin{array}{c} U \text{ is an open connected neighborhood of 0 in } \mathbb{R}, \\ \gamma \text{ is an integral curve of } v \text{ and } \gamma(0) = x_0 \end{array} \right\}.$$

This partially ordered set has a greatest element; let $V = \bigcup\{U \mid \text{there is some } (U, \gamma) \in Z\}$, we have $\gamma : V \to M$, apply Zorn's lemma as usual, etc. ☐

Examples.

- Let $M = \mathbb{R}^n$, and let v be a constant vector field. Then for any $x \in M$, the curve $\gamma : \mathbb{R} \to M$ defined by $\gamma(t) = x + tv$ is an integral curve.
- Let $M = \mathbb{R}^n$, and let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Define the vector field $v(x) = Sx$ for all $x \in \mathbb{R}^n$. As you've seen in Victor Ginzburg's class, for any $A \in \mathrm{M}_n(\mathbb{R})$ we can define

  $$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

  which satisfies $\exp(A + B) = \exp(A) \exp(B)$ when $AB = BA$, and $\|\exp(A)\| \leq \exp(\|A\|)$. In particular, $\exp(A) \in \mathrm{GL}_n(\mathbb{R})$, and $\exp((t_1 + t_2)A) = \exp(t_1A) \exp(t_2A)$.

  The integral curve of $v(x) = Sx$ with $\gamma(0) = x$ is simply the curve $\gamma(t) = (\exp(tS))x$.

Proof of existence and uniqueness. We use the contraction principle, which states that if X is a complete metric space and $0 \leq \lambda < 1$, and $T : X \to X$ is a continuous function such that $d(Tx_1, Tx_2) \leq \lambda d(x_1, x_2)$ for all $x_1, x_2 \in X$, then there is a unique fixed point of T (this follows simply by observing that $\{T^n x\}_{n \in \mathbb{N}}$ is a Cauchy sequence for any $x \in X$).

WLOG, let $x_0 = 0$, and let $M =$ an open set $\Omega \subseteq \mathbb{R}^n$ which is a neighborhood of 0. We have our C^1 vector field v on M.

We want to construct a $\gamma : (-c, c) \to \Omega$ such that $\gamma(0) = 0$ and such that $\gamma'(t) = v(\gamma(t))$ for all $t \in (-c, c)$. Thus, we want

$$\gamma(t) = \int_0^t v(\gamma(t))\, dt.$$

Let $r > 0$ such that $\overline{B_r(0)} = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subset \Omega$.

Let X be the space of continuous functions $[-c, c] \to \overline{B_r(0)}$ such that $\gamma(0) = 0$. We give X the metric induced by the sup norm.

For any $\gamma \in X$, define

$$(T\gamma)(t) = \int_0^t v(\gamma(t))\, dt.$$

We want T to be a well-defined map $X \to X$, and to be a contraction.

Because $\overline{B_r(0)}$ is compact and v is continuous, we have that $v(\overline{B_r(0)}) \subseteq B_C(0)$ for some C.

(i) By taking c to be sufficiently small, we have $cC \leq r$. This implies that T is a well-defined map $T : X \to X$, because $t \in [-c, c]$ and $v(\gamma(t)) \in \overline{B_r(0)}$ implies that

$$|(T\gamma)(t)| = \left| \int_0^t v(\gamma(t))\, dt \right| \leq cC \leq r,$$

so that $T\gamma \in X$.

(ii) Now we want to show that T is a contraction. For any $\gamma_1, \gamma_2 \in X$, we have

$$|(T\gamma_1)(t) - (T\gamma_2)(t)| = \left| \int_0^t v(\gamma_1(h)) - v(\gamma_2(h))\, dh \right| \leq \int_0^t \|v'\|_\infty \|\gamma_1(h) - \gamma_2(h)\|\, dh \leq cM\|\gamma_1 - \gamma_2\|.$$

By shrinking c further if necessary, we can make $cM \leq \lambda < 1$, where M is defined by

$$M = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| \leq r}} \{\|v'(x)\|\},$$

where $\|v'(x)\|$ denotes the operator norm of $v'(x) : \mathbb{R}^n \to \mathbb{R}^n$. $\square$

Lecture 17 (2013-02-15)

Today we'll talk about a method for reducing nth order differential equations to first order differential equations. Thus, for example, we can express acceleration as a function of position and velocity.

Let $\Omega \subset \mathbb{R}^n$ be open, and let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function (representing acceleration). Then given $(x_0, y_0) \in \Omega \times \mathbb{R}^n$, there is a unique $\gamma : (-\epsilon, \epsilon) \to \Omega$ such that $\gamma(0) = x_0$, $\gamma'(0) = y_0$, and $\gamma''(t) = a(\gamma(t), \gamma'(t))$ for all $t \in (-\epsilon, \epsilon)$.

The trick was to introduce the space $\Omega \times \mathbb{R}^n$. Put $M = \Omega \times \mathbb{R}^n$, and let $\delta(t) = (\gamma(t), \gamma'(t))$. Any vector field on M can be thought of as a function $M \to \mathbb{R}^n \times \mathbb{R}^n$; let $\widetilde{v} : M \to \mathbb{R}^n \times \mathbb{R}^n$ be the vector field defined by

$$\widetilde{v}(x, y) = (y, a(x, y)).$$

If γ is as above, then we have $\delta(0) = (x_0, y_0) \in M$, and $\delta'(t) = (\gamma'(t), \gamma''(t)) = \widetilde{v}(\delta(t))$. Thus, curves γ satisfying our requirements are precisely integral curves of the vector field $\widetilde{v}$ on M.

Theorem. *Let M be a C^∞ manifold and let v be C^∞ vector field on M. Let $p \in M$. Then there is a neighborhood $U(p)$ of p in M, some $a > 0$, and a C^∞ function $\gamma : U(p) \times (-a, a) \to M$ such that for all $x \in U(p)$, $\gamma_x(t)$ is the (unique) integral curve of v such that $\gamma_x(0) = x$, where $\gamma_x(t) := \gamma(x, t)$.*

A reference for this theorem is Hurewicz's *Lectures on Ordinary Differential Equations.*

Remark 1. Let γ be an integral curve of a C^∞ vector field. Then it is clear that γ is C^∞, because we have that

$$\gamma'(t) = v(\gamma(t)),$$

so that if γ is C^k, then $v \circ \gamma$ is C^k, hence γ' is C^k, so that γ is C^{k+1}.

Remark 2. Assume that v is a C^1 vector field. The contraction principle shows that $(x, t) \mapsto \gamma_x(t)$ is continuous in (x, t) and defined on some neighborhood $U(p) \times (-a, a)$, as follows:

WLOG, let $p = 0$, and let $M = \Omega$, an open subset of $\mathbb{R}^n$. Then the space of continuous functions

$$\{x \in \mathbb{R}^n \mid \|x\| \le \alpha\} \times [-c, c] \quad \longrightarrow \quad \{x \in \mathbb{R}^n \mid \|x\| \le R\}$$

with the $\|\cdot\|_\infty$ norm is a complete metric space.

Remark 3. Assume that v is a C^2 vector field on Ω, an open subset of $\mathbb{R}^n$. We want to show that the partial derivatives

$$\frac{\partial}{\partial x_i}(\gamma_x(t))$$

exist. We can simply check that the differential equation defining them is satisfied. Let $v : \Omega \to \mathbb{R}^n$ be a vector field on Ω, and so that for any $x \in \Omega$ we have $v'(x) : \mathbb{R}^n \to \mathbb{R}^n$.

We introduce the (standard) notation $\phi_t(x) = \gamma_x(t)$. Thus, ϕ_t is defined at all $t \in (-a, a)$, and ϕ_t is a function $\phi_t : U(p) \to \Omega$, so that we have $\phi_t'(x) : \mathbb{R}^n \to \mathbb{R}^n$ for any $x \in U(p)$, where $\phi_t'(x)$ is the derivative of $x \mapsto v(\phi_t(x))$.

Let $\widetilde{v} : M \to \mathbb{R}^n \times \mathrm{M}_n(\mathbb{R})$ be the vector field on $M = \Omega \times \mathbb{R}^n$ defined by

$$\widetilde{v}(x, S) = (v(x), v'(x)S).$$

We claim that, for any fixed t, the map $x \mapsto (\phi_t(x), \phi_t'(x))$ is an integral curve for $\widetilde{v}$.

We have that

$$\frac{d}{dt}\phi_t(x) = v(\phi_t(x)),$$

so that

$$\widetilde{v}(\phi_t(x), \phi_t'(x)) = \frac{d}{dt}(\phi_t(x), \phi_t'(x)) = (v(\phi_t(x)), v'(\phi_t(x))\phi_t'(x)).$$

This "reduces" the question to integral curves of $\widetilde{v}$ on $\Omega \times \mathrm{M}_n(\mathbb{R})$ (?).

Corollary. *Let M be a C^∞ manifold, and let v be a C^∞ vector field on M. For all $x \in M$, we've shown that there is a maximal integral curve $\gamma_x : I_x \to M$ such that $\gamma_x(0) = x$. Let*

$$D = \{(x,t) \in M \times \mathbb{R} \mid t \in I_x\},$$

and write $\gamma_x(t) = \gamma(x,t)$ for all $(x,t) \in D$.

1. *D is an open subset of $M \times \mathbb{R}$.*
2. *If $\phi_t(x)$ is defined, then $\phi_s(x)$ is defined for all $0 \leq s \leq t$ (or all $t \leq s \leq 0$).*
3. *If $\phi_t(x)$ is defined and if $\phi_s(\phi_t(x))$ is defined, then $\phi_{t+s}(x)$ is defined and*
$$\phi_{t+s}(x) = \phi_s(\phi_t(x)).$$
4. *Assume M is compact. Then the theorem implies that there is an open interval $(-a,a)$ such that $(-a,a) \subset I_x$ for all $x \in M$, so that $D \supseteq M \times (-a,a)$; statement 3 then implies $D = M \times \mathbb{R}$. Furthermore, for a fixed t, the map $x \mapsto \phi_t(x)$ is C^∞ and induces an isomorphism on tangent spaces.*
5. *Observe that given a vector field v on M, if a constant path $\gamma(t) = p$ is an integral curve of v, then $v(p) = 0$. Conversely, if $v(p) = 0$, then the constant curve to p is an integral curve.*

Proof of 3. We know γ_x is defined on $[0, t+\epsilon)$, and that $\gamma_{\phi_t(x)}$ is defined on $[0, s+\epsilon')$. Now define

$$\delta(z) = \begin{cases} \gamma_x(z) & \text{for all } z \in [0,t], \\ \gamma_{\phi_x(t)}(z-t) & \text{for all } z \in [t, s+\epsilon). \end{cases}$$

Note that δ is an integral curve of v.

□

Exercise. Let $\mathbb{R} \xrightarrow{h} \mathrm{Diffeo}(M)$ be a group homomorphism such that the map $M \times \mathbb{R} \to M$ defined by $(x,t) \mapsto h(t)(x)$ is C^∞. Prove that there is a C^∞ vector field v on M such that $h(t) = \phi_t$ for the vector field v. This does not require M to be compact, just that the support of v is compact.

Remark. If v is a vector field on a manifold M and $v(p) \neq 0$, then there is a neighborhood $U(p)$ of p in M and a diffeomorphism $f : U(p) \to \Omega$, where Ω is an open subset of $\mathbb{R}^n$, such that the vector field v (restricted to $U(p)$) is carried by f to the constant vector field $\frac{\partial}{\partial x_1}$ on Ω.

Proof of Remark. Let's say that Z is a slice of M when $T_pZ \oplus \mathbb{R}v(p) = T_pM$. The hypotheses of the inverse function theorem hold at $(p,0)$, so that we can find coordinates in which the map from $Z \times (-\epsilon, \epsilon) \to M$ defined by $\phi_t(z) = \gamma_z(t)$ sends the vector field v to a constant vector field, such as for example $\frac{\partial}{\partial x_1}$. □

Next time, we'll start with Lie brackets. It's motivated from two points of view; maybe 100 in fact.

Lecture 18 (2013-02-18)

Though it'll seem like we're leaving integral curves, we'll return to them in the middle of the lecture.

Recall that given a C^∞ manifold M, a point $p \in M$, and a tangent vector $v \in T_pM$, there is an $\mathbb{R}$-linear functional $v : C^\infty(M) \to \mathbb{R}$, sending a C^∞ function $f : M \to \mathbb{R}$ to $v(f) \in \mathbb{R}$. It satisfies the Leibniz rule,

$$v(fg) = f(p)v(g) + g(p)v(f).$$

This is a generalization of the notion of directional derivative in Euclidean space.

Now let v be a vector field on M. Let $R = C^\infty(M)$. Now we have an $\mathbb{R}$-linear map $v : R \to R$, defined by $v(f)(p) = v(p)(f)$ for all $p \in M$. For example, if $M = \mathbb{R}^n$ and $v = (a_1, \dots, a_n) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we have that

$$v(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

For any ring S, a function $D : S \to S$ is a derivation when $D(fg) = D(f) \cdot g + f \cdot D(g)$. Very often, we are given a subring $T \subset S$ contained in the center of S, that we require to satisfy $D(t) = 0$ for all $t \in T$. Note that the map $v : R \to R$ sending $f \mapsto v(f)$ is a derivation, and $v(\text{any constant function}) = 0$ (observe that we can $\mathbb{R}$ is a subring of R).

The following is an easy lemma.

Lemma. *If $D_1, D_2 : R \to R$ are derivations, then $(D_1 \circ D_2) - (D_2 \circ D_1)$ is also a derivation.*

In particular, if v, w are C^∞ vector fields on M, U is an open subset of M, and $C^\infty(U)$ is the ring of C^∞ functions on U, the map $f \mapsto v(w(f)) - w(v(f))$ is a derivation of $C^\infty(U)$. If we fix a point $p \in M$, we can consider neighborhoods U of $p \in M$, and the map

$$f \mapsto (v(w(f)) - w(v(f)))(p)$$

induces an $\mathbb{R}$-linear map on germs $C^\infty_{M,p} \to \mathbb{R}$. Being a derivation, this is equal to $h(p)(f)$ for a unique $h(p) \in T_pM$. It is true (though we won't check) that $p \mapsto h(p)$ is a C^∞ vector field on M, and we define the Lie bracket of v and w to be this h. We write $h = [v, w]$. Thus,

$$[v, w](f) = v(w(f)) - w(v(f))$$

for all C^∞ maps $f : U \to \mathbb{R}$.

Lemma. *Let Ω be an open subset of $\mathbb{R}^n$, and let v, w be C^∞ vector fields on Ω. Then*

$$[v, w] = D_v w - D_w v,$$

where

$$(D_v w)(x) = \frac{d}{dt} w(x + tv)\bigg|_{t=0}.$$

Proposition. *The $\mathbb{R}$-vector space of C^∞ vector fields on M, together with the bracket, satisfies the axioms of a Lie algebra:*

1. *$[v, w] = -[w, v]$ for all C^∞ vector fields v and w.*
2. *$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$ for all C^∞ vector fields v_1, v_2, v_3.*
3. *$[tv, w] = t[v, w]$ for all $t \in \mathbb{R}$.*

Definition. Let M and N be C^∞ manifolds, and let $\phi : M \to N$ be a C^∞ map. Given vector fields v on M and w on N, we say that v and w are ϕ-related if for all $x \in M$,

$$\phi'(x)v(x) = w(\phi(x)).$$

Lemma 1. *Given vector fields v on M and w on N, they are ϕ-related if and only if $\phi(\gamma)$ is an integral curve of w for any integral curve γ of v.*

Proof. Assume that v and w are ϕ-relted. Let $\gamma : (a,b) \to M$ be an integral curve for v, so that for all $t \in (a,b)$, we have

$$\gamma'(t) = v(\gamma(t)).$$

Let $\delta = \phi \circ \gamma$. Then

$$\delta'(t) = \phi'(\gamma(t))\gamma'(t) = w(\delta(t)).$$

Everything is reversible, so we are done. □

Last time, I mentioned that if a vector field is non-zero at a point, then in some neighborhood it looks like $\frac{\partial}{\partial x_1}$. There is a proof of this in Warner's book on page 40.

Example. Let w be a vector field on N and suppose that $w(p) \neq 0$. Then there is a chart centered at p such that w is transformed to $\frac{\partial}{\partial x_n}$.

Proof. Let Zbe a codimension 1 closed submanifold of N containing p, and suppose that it is transverse, i.e. that $T_pZ \oplus \mathbb{R}w(p) = T_pN$. Let $\delta_y(t)$ be an integral curve of w with initial value y, i.e. $\delta_y(0) = y$. Let $M = Z \times (-c, c)$, and let $\phi : M \to N$ be the map defined by

$$\phi(z,t) = \delta_z(t).$$

This is a diffeomorphism in a neighborhood of $Z \times \{0\}$ by the inverse function theorem, and the curves $t \mapsto (z,t)$ on M are sent by ϕ to the curves $\delta_z(t)$ on N, which are integral curves of w. Thus, $t \mapsto (z,t)$ is an integral curve for $\frac{\partial}{\partial x_n}$. □

Lemma 2. *Let M and N be C^∞ manifolds, and let $\phi : M \to N$ be C^∞.*

(a) If v on M and w on N are ϕ-related, then $v(\phi^* f) = \phi^* w(f))$ for any C^∞ map $f : N \to \mathbb{R}$; this is just a restatement of the definition.

(b) If v_1 is ϕ-related to w_1 and v_2 is ϕ-related to w_2, then $[v_1, v_2]$ and $[w_1, w_2]$ are ϕ-related.

Proof of (b). We have

$$v_1(v_2(\phi^*(f))) = v_1(\phi^*(w_2(f))) = \phi^*(w_1(w_2(f))).$$

Now interchange and subtract. □

Remark. This has an important consequence. If M is a locally closed submanifold of N, $\phi : M \to N$ is the inclusion, and w is a vector field on N, then to say that there is some v on M that is ϕ-related to w is equivalent to saying that $w(x) \in T_xM$ for all $x \in M$ (because $w(x) = v(x)$). Thus, Lemma 2 is saying something about vector fields that are tangent to submanifolds; if w_1 and w_2 are vector fields on N such that $w_1(x), w_2(x)$ belong to T_xM for all $x \in M$, then $[w_1, w_2]$ has the same property.

Definition. Let M be a C^∞ manifold, and let W be a C^∞ subbundle of TM of rank r. A locally closed submanifold A of M is a leaf if for all $x \in A$, $T_xA = W(x)$.

(The notion of leaf can defined in more generality than what is given here.)

Suppose that there is a leaf of W through every point of M. If w_1, w_2 are C^∞ sections of W, then $[w_1, w_2]$ is necessarily also a section of W; we can see this easily as follows. Let $p \in M$ and let Z be a leaf through p. Because Z is a leaf, w_1 and w_2 are tangential to Z, so $[w_1, w_2]$ is tangential to Z, i.e. $[w_1, w_2](p) \in T_pZ = W(p)$ for all $p \in M$.

Definition. A C^∞ subbundle W of TM is said to be involutive (alternatively, integrable) if for all C^∞ sections w_1, w_2 of W, $[w_1, w_2]$ is also a section of W.

We have already proven one piece of the following theorem:

Theorem (Frobenius)**.** *Let W be a subbundle of TM. The following are equivalent:*

1. *W is involutive.*
2. *There is a leaf of W through every point.*
3. *For all $p \in M$, there is a diffeomorphism h from a neighborhood of p to $U_1 \times U_2$, where U_i is an open subset of $\mathbb{R}^{n_i}$ for $i = 1, 2$, such that $h(W)$ is the constant $\mathbb{R}^{n_1} \times \{0\}$ bundle on $U_1 \times U_2$.*

Proof. It is clear that $3 \implies 2$, and we have already proven that $2 \implies 1$, so it remains to prove that $1 \implies 3$. This proof is taken from Narasimhan (the proof is originally due to Volterra).

Step 1. Let W be an involutive subbundle of rank r. Then in a neighborhood of any $p \in M$, we can find vector fields $w_1, \ldots, w_r$ which are a frame for W, i.e. $w_1(x), \ldots, w_r(x)$ are a basis for $W(x)$ for all x in the neighborhood, and such that $[w_i, w_j] = 0$ for all i, j.

Let me make a linear algebra observation: given a vector space $V = V_1 \oplus V_2$, subspaces $W \subset V$ such that the projection to V_1 is an isomorphism, i.e.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & V \arrow[r, "p_1"] & V_1 \end{tikzcd}$$

can be identified with graphs of linear transformations $S : V_1 \to V_2$.

Now write $\mathbb{R}^N = V_1 \times V_2$, where $N = \dim(M)$, where V_1 and V_2 have been chosen such that $p|_{W(p)} : W(p) \to V_1$ is an isomorphism (p is the projection $\mathbb{R}^N \to V_1$), so that $W(x) \cong V_1$ for all x in some neighborhood of p. Thus, for each x, we get $S(x) : V_1 \to V_2$, and

$$W(x) = \{(v_1, S(x)v_1) \mid v_1 \in V_1\}.$$

Let $\Omega \subset V_1 \times V_2 = \mathbb{R}^N$ be open. WLOG we have $V_1 = \mathbb{R}^r$, where $e_1, \ldots, r_r$ are the standard basis of $\mathbb{R}^r$. We have $S(x)e_i = u_i(x)$, where $u_i : \Omega \to V_2$ is some C^∞ function. Thus $W(x)$ is the linear space of the $e_i + u_i$. For any i, j, we have that $[e_i + u_i, e_j + u_j]$ is a section of W, and using the formula

$$[\alpha, \beta] = D_\alpha \beta - D_\beta \alpha$$

on Euclidean space, we have that $[e_i + u_i + e_j + u_j]$ is a section of V_2 (i.e. a function $\Omega \to V_2$); but it also has to be a section of W, so it has to be 0 since $V_2 \cap W(x) = 0$ for all $x \in \Omega$. $\square$

We'll finish the proof of this with Step 2 next time.

Lecture 19 (2013-02-20)

Everything we're talking about today will be C^∞.

To finish the proof of the Frobenius theorem from last time, it remains to show the following result:

Lemma 1. *If $w_1, \ldots, w_r$ are linearly independent, commuting vector fields (commuting in the sense that their pairwise Lie brackets are 0), then there is a chart centered at any given point where the w_i are transformed to the coordinate vector fields $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, r$.*

Remark. Let v and w be vector fields on M. Let ϕ_t and ψ_s denote the one-parameter groups for v and w respectively (i.e. the flows). Then for all $p \in M$, there is some neighborhood $U(p)$ of p and $(-\epsilon, \epsilon)$ such that $\phi_t(\psi_s(x))$ and $\psi_s(\phi_t(x))$ are defined for all $x \in U(p)$ and $t, s \in (-\epsilon, \epsilon)$.

Lemma 2. *With notation as above, if $[v, w] = 0$, then $\phi_t(\psi_s(x)) = \psi_s(\phi_t(x))$ for any $x \in U(p)$ and $s, t \in (-\epsilon, \epsilon)$.*

Proof that Lemma 2 $\implies$ Lemma 1. Let's assume the result of Lemma 2 in the case that $v(p) \neq 0$. Let ϕ_t^i denote the one-parameter groups with respect to w_i for each $i = 1, \ldots, r$. Let $p \in M$, and select a locally closed C^∞ submanifold $Z \subset M$ with $p \in Z$ such that $T_pZ \oplus \mathbb{R}w_1(p) \oplus \cdots \oplus \mathbb{R}w_r(p) = T_pM$. Note that by assuming this is true at p, we can assume this is true in a neighborhood of p.

Let $h : (-\epsilon, \epsilon)^r \times (Z \cap U(p)) \to M$ be defined by

$$h(x_1, \ldots, x_r, z) = \phi_{x_1}^1 \phi_{x_2}^2 \cdots \phi_{x_r}^r(z).$$

We see that h induces an isomorphism from the tangent space at $(0, \ldots, 0, z)$ to T_zM for all $z \in Z \cap U(p)$. Note that $h(t, x_2, \ldots, x_r, z)$ is an integral curve for w_1, so that $h'(?)\frac{\partial}{\partial x_1} = w(h(?))$ for all ? in the domain of h (this is not a ? in the sense of "I didn't get down what was on the board", but rather "?" itself what was written on the board). This is

$$\phi_{x_2}^2 \phi_{x_1}^1 \cdots,$$

and thus we see that $h'(?)\frac{\partial}{\partial x_2} = w_2(h(?))$, etc. (not sure I understand this part). □

Proof of Lemma 2. We have that $w_1(p) \neq 0$, so (as we have shown earlier) we can assume WLOG that $w = \frac{\partial}{\partial x_1}$. For any vector $v = \sum a_i \frac{\partial}{\partial x_i}$, we have that

$$[w, v] = \sum \frac{\partial a_i}{\partial x_1} \cdot \frac{\partial}{\partial x_i}.$$

By assumption, this is zero, so the a_i's are (in some neighborhood) functions of $(x_2, \ldots, x_n)$. Because the statement is local, we can assume that we are working on $(-\epsilon, \epsilon) \times \Omega$ for an open subset $\Omega \subset \mathbb{R}^{n-1}$. Let $c \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. Let $h_c : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \to (-\epsilon, \epsilon) \times \Omega$ be defined by

$$h_c(x_1, x_2, \ldots) = (x_1 + c, x_2, \ldots).$$

Then v and $v|_{(-\frac{\epsilon}{2}, \frac{\epsilon}{2})}$ are h_c-related.

Therefore, if δ is an integral curve of v, then $h_c \circ \delta$ is also an integral curve. Let ϕ_t denote the one-parameter group associated to v. Then we have that

$$h_c \circ \phi_t = \phi_t \circ h_c.$$

But $h_c = \psi_c$ where ψ_c is the one-parameter group associated to w. □

Theorem (Thom's ambient isotopy lemma)**.** *Let $I = [0,1]$, let A and B be C^∞ manifolds where A is compact, and let $F : A \times I \to B$ be a C^∞ map. Let $f_t : A \to B$ be defined by $f_t(a) = F(a,t)$ for all $a \in A$ and $t \in [0,1]$. If f_t is an embedding for all $t \in I$, then there is a C^∞ map $G : B \times I \to B$ such that g_t is a diffeomorphism for all $t \in I$, and $f_t = g_t \circ f_0$ for all $t \in I$, where $g_t(b) = G(b,t)$.*

Recall that if A is an arbitrary subset of a C^∞ manifold M, then given a map $f : A \to \mathbb{R}$, we say that it is C^∞ map when there exist open sets $U_\lambda \subset M$ for all $\lambda \in \Lambda$ such that $f|_{A\cap U_\lambda} = f_\lambda|_{A\cap U_\lambda}$ and $W := \bigcup U_\lambda$ contains A. Then $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open cover of W, so there is a partition of unity subordinate to this cover. Let $\varphi_\lambda : W \to \mathbb{R}$ be subordinate to U_λ.

Consider $\varphi_\lambda|_{U_\lambda} \circ f_\lambda : U_\lambda \to \mathbb{R}$, which has support contained in U_λ, and extends by zero to a C^∞ function on W denoted by $\varphi_\lambda f_\lambda$. If we then define $\widetilde{f} = \sum_{\lambda\in\Lambda} \varphi_\lambda f_\lambda$, then $\widetilde{f}$ is a C^∞ function defined on W that extends f. More generally, if we have a C^∞ bundle

$$\begin{tikzcd}
& V \arrow[d, "p"] \\
A \arrow[ur, "s"] \arrow[r, hook] & M
\end{tikzcd}$$

where A is arbitrary, then what we've shown is that it extends to a C^∞ section on an open $W \supset A$.

A variant of this result is that if A is a closed set, then note that $\{U_\lambda \mid \lambda \in \Lambda\} \cup \{M - A\}$ is also an open cover, so we can create a partition of unity $\{\varphi_\lambda \mid \lambda \in \Lambda\} \cup \{\varphi_0\}$. If we define $f_0 : (M - A) \to \mathbb{R}$ to be zero, then let

$$\widetilde{f} = \sum_\lambda \varphi_\lambda f_\lambda + \varphi_0 f_0.$$

Once again, $\widetilde{f} : M \to \mathbb{R}$ and $\widetilde{f}|_A = f$. Finally, if A is compact, then we see that $\widetilde{f}$ can be chosen to have compact support.

Proof of Thom's lemma. WLOG, we can assume that $B \subset \mathbb{R}^N$, so that $F : A \times I \to B$ can be extended to a C^∞ map $F : A \times (-\epsilon, 1+\epsilon) \to B$. This is because we can extend to a map $A \times \mathbb{R} \to \mathbb{R}^N$, and letting U be a tubular neighborhood around B in $\mathbb{R}^N$, we can find an open neighborhood V around $A \times I$ in $A \times \mathbb{R}$ that maps into U, and because A is compact we can take V to be of the form $A \times (-\epsilon, 1+\epsilon)$, and then we can use the retraction from U to B to map everything into B.

$$\begin{tikzcd}
A \times I \arrow[d] \arrow[r, phantom, "\subset"] & V \arrow[d] \arrow[r, phantom, "\subset"] & A \times \mathbb{R} \arrow[d] \\
B \arrow[r, phantom, "\subset"] & U \arrow[r, phantom, "\subset"] & \mathbb{R}^N
\end{tikzcd}$$

Because A is compact, we can assume that f_t is an embedding for all $t \in (-\epsilon, 1+\epsilon)$. Define $\widetilde{F} : A \times (-\epsilon, 1+\epsilon) \to B \times (-\epsilon, 1+\epsilon)$ to be the map sending $(a,t) \mapsto (F(a,t), t)$. Then $\widetilde{F}$ sends $(0, \frac{d}{dt})$ to a vector field $(w, \frac{d}{dt})$. Let $C = \widetilde{F}(A \times (-\epsilon, 1+\epsilon))$.

We have that $C \hookrightarrow B \times (-\epsilon, 1+\epsilon)$ is closed and a section w of $p_1^*TB|C$, where $p_1 : B \times (-\epsilon, 1+\epsilon) \to B$. There exists a global C^∞ section $\widetilde{w}$ that extends w. Consider $v = (\widetilde{w}, \frac{d}{dt})$, which is a vector field on $B \times (-\epsilon, 1+\epsilon)$. Let ϕ_t be the flow associated to v.

Fact 1: We know that for all $a \in A$, the map $t \mapsto (f_t(a), t)$ is an integral curve.

Fact 2: We may assume that $\operatorname{supp}(\widetilde{w}) \xrightarrow{p_2} (-\epsilon, 1+\epsilon)$ is proper. This implies that for all $z \in (-\epsilon, 1+\epsilon)$, the flow $\phi_t(B \times z)$ is defined for all t with $|t| < \delta$, say. In particular, $\phi_t(B \times z)$ is defined for all $z \in I$ and fro all t with $|t| < \delta$.

Fact 3: We have that $\phi_t(B \times z) \subset B \times \{z + t\}$, from which it follows that for all $0 \leq z \leq 1$, ϕ_t is defined on $B \times z$ for all $-z \leq t \leq 1 - z$.

From these facts, we have that $\phi_t|_{B\times 0} \xrightarrow{\cong} B \times t$ is a diffeomorphism for all $0 \leq t \leq 1$. Now define $g_t = \phi_t$ and we are done. $\square$

Lecture 20 (2013-02-22)

Lecture 21 (2013-02-25)

Lecture 22 (2013-02-27)

Let M be a C^∞ manifold and v a C^∞ vector field on M. Let $\phi_t(x) = \gamma_x(t)$ be the integral curve for v with $\gamma_x(0) = x$. Let ω be any object attached to the manifold, such as for example a section of $TM^{\otimes m} \otimes T^*M^{\otimes n}$. Then the Lie derivative of ω with respect to v makes sense:

$$L_v\omega = \frac{d}{dt}\phi_t^*\omega\bigg|_{t=0}$$

In particular, $L_v w$ is defined when w is a vector field.

Proposition. *For all vector fields v, w on M, we have $L_v w = [v, w]$.*

Lemma (Leibniz rule for sections of bundles)**.** *Let v be a vector field.*

1. *$L_v(\omega \wedge \eta) = (L_v\omega) \wedge \eta + \omega \wedge L_v(\eta)$, where ω is a k-forma and η is an ℓ-form*
2. *$L_v i_w\theta = i_{L_v(w)}\theta + i_w L_v\theta$¡ where w is a vector field, and θ is a k-form*
3. *$v(\theta(w)) = \theta(L_v(w) + (L_v\theta)(w)$, where θ is a 1-form (this is just a special case of 2)*

Proof. Let V_1, V_2, V_3 be vector bundles on M, and let B be a bilinear map

$$\begin{tikzcd}
V_1 \times_M V_2 \arrow[rr, "B"] \arrow[dr] & & V_3 \arrow[dl] \\
& M &
\end{tikzcd}$$

i.e. $B(x) : V_1(x) \times V_2(x) \to V_3(x)$ is bilinear for all $x \in M$. Let s_t^1, s_t^2 be families of C^∞ sections of V_1 and V_2 respectively, indexed by $t \in (-\epsilon, \epsilon)$. Let p_1 be the projection $p_1 : M \times (-\epsilon, \epsilon) \to M$, so that each s_i is a secton of $p_1^*V_i$. Then

$$\frac{d}{dt}B(s_t^1, s_t^2) = B\left(\frac{d}{dt}\,s_t^1, s_t^2\right) + B\left(s_t^1, \frac{d}{dt}\,s_t^2\right).$$

How will we apply this - we want to choose $s_i = \phi_t^*(?)$.

Let $V_1 = TM$, $V_2 = \Lambda^k T^*M$, $V_3 = \Lambda^{k-1}T^*M$, and let $B(x) : T_xM \times \Lambda^k T_x^*M \to \Lambda^{k1-}T_x^*M$ be defined by $B(x)(\omega, \theta) = i_\omega(\theta)$.

Part 2 is then an application of the Leibniz rule

$$i_{v^*}(\omega \wedge \eta) = i_{v^*}(\omega) \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge i_{v^*}(\eta)$$

where $v^* \in V^*$ and $\omega \in \Lambda^k V$, and 3 is just 2 for $k = 1$.

Given $\theta = df$, where $f : M \to \mathbb{R}$ is a C^∞ map, then

$$\theta(w) = (df)(w) = w(f)$$

That $v(\theta(w)) = v(w(f))$ is just the left side of 3. But

$$\theta(L_v w) = (L_v w)(f),$$

hence

$$L_v(\theta) = L_v(df) = dL_v f = d(v(f)),$$

hence

$$(L_v\theta)(w) = w(v(f)).$$

Now 3 reads as

$$v(w(f)) = w(v(f)) + (L_v w)(f),$$

i.e.

$$(L_v w)(f) = v(w(f)) - w(v(f)) = [v, w](f)$$

for all C^∞ maps $f : M \to \mathbb{R}$. ☐

Corollary (Special case of Cartan's formula)**.** *Let ω be a 1-form, and let v_1 and v_2 be vector fields. Then*

$$d\omega(v_1, v_2) = v_1(\omega(v_2)) - v_2(\omega(v_1)) - \omega([v_1, v_2])$$

Remark. Note that we can identify $\Lambda^k T_x^* M$ with $(\Lambda^k T_x M)^*$ as follows: given $\omega \in \Lambda^k T_x^* M$, we define

$$\omega(v_1, v_2, \ldots, v_k) = i_{v_k} i_{v_{k-1}} \cdots i_{v_1} \omega \in \Lambda^k T_x^* M \in \mathbb{R}$$

for $v_1, \ldots, v_k \in T_x M$.

Proof. We have that $L_v = i_v d + d i_v$. Thus,

$$i_{v_1} d\omega = L_{v_1}\omega - d(i_{v_1}\omega),$$

so that

$$\begin{aligned} d\omega(v_1, v_2) = (i_{v_1} d\omega) v_2 &= (L_{v_1}\omega) v_2 - v_2(\omega(v_1)) \\ &= L_{v_1}(\omega(v_2)) - \omega(L_{v_1} v_2) - v_2(\omega(v_1)) \\ &= v_1(\omega(v_2)) - \omega([v_1, v_2]) - v_2(\omega(v_1)). \end{aligned}$$ ☐

Remark. We defined

$$L_v\omega = \frac{d}{dt}\,\phi_t^*\omega\bigg|_{t=0}.$$

It is more generally true that

$$\frac{d}{dt}\,\phi_t^*(\omega)\bigg|_{t=t_0} = \phi_{t_0}^*(L_v\omega).$$

Note that we haven't said what kind of thing ω is; it only makes sense for certain natural bundles. But this works in particular when ω is some vector field w. Then ϕ_t is the flow associated to v; also, let ψ_s be the flow associated to w. Then

$$[v, w] = 0 \iff L_v(w) = 0 \iff \frac{d}{dt}(\phi_t^* w) = 0 \text{ for all } t \iff \phi_t^* w = w \text{ for all } t.$$

Assume that $[v, w] = 0$, so that $\phi_t^* w = w$, and thus ϕ_t(integral curve of w) is an integral curve of w. This is equivalent to saying that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$. Thus, we have established the following:

Corollary. $[v, w] = 0 \iff \phi_t \circ \psi_s = \psi_s \circ \phi_t$

Theorem (Ehresmann's theorem)**.** *Let $f : X \to Y$ be a proper submersion. Then f is a C^∞ fiber bundle.*

We will give a second proof of this using flows.

Proof. Let $X \hookrightarrow \mathbb{R}^m$ be an embedding of X in Euclidean space. Thus, for any $x \in X$, T_xX gets an inner product. We have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T_x f^{-1}(f(x)) \arrow[r] & T_xX \arrow[r, "f'(x)"] & T_{f(x)}Y \arrow[r] & 0 \end{tikzcd}$$

where we have used that f is a submersion. Let $W(x) = T_x f^{-1}(f(x))^{\perp}$, so that we get a subbundle W of TX such that $f'(x) : W(x) \xrightarrow{\cong} T_{f(x)}Y$.

Assume that $Y = (-1,1)^n \subset \mathbb{R}^n$. Thne $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ are vector fields on Y, i.e. sections of TY, and so we get corresponding sections $w_1, \dots, w_n$ of W such that $f'(x)(w_i) = \frac{\partial}{\partial y_i}$ for all i. Note that even though the $\frac{\partial}{\partial y_i}$ all commute with each other, we need not have that the w_i all commute with each other.

Let ϕ^i_t denote the flow associated to w_i. One sees that for any compact $K \subseteq X$, there is an $\epsilon > 0$ such that $\phi^1_{t_1} \cdots \phi^n_{t_n}(x)$ are defined for all $x \in K$ and $t_i \in (-\epsilon, \epsilon)$.

Let $K = f^{-1}(0)$, which is compact because f is proper. Then we have a commutative diagram

$$\begin{tikzcd} K \times (-\epsilon,\epsilon)^n \arrow[r, "h"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ (-\epsilon,\epsilon)^n \arrow[r, hook] & (-1,1)^n \end{tikzcd}$$

and h induces isomorphisms on tangent spaces at $K \times 0$, so it must do so in a neighborhood of $K \times 0$. Because p_2 is proper, it follows that by shrinking ϵ if necessary, we may assume that h induces isomorphisms on tangent spaces everywhere, and that h is one-to-one. Then h is then a diffeomorphism onto its image U, which is open in X. We want to show that $U = X$; thus, let $F = X \setminus U$. Then F is closed in X, and because f is proper, we have that $f(F)$ is a closed set (we're using Hausdorffness here). Then $F \cap f^{-1}(0) = \varnothing$, because $0 \notin f(F)$, and now replace $(-1,1)^n$ by the complement of $f(F)$. ☐

We can now state a refinement of Ehresmann's theorem.

Theorem. *Let $f : X \to Y$ be a proper submersion, and let $A \subseteq X$ be a closed C^∞ submanifold. Assume also that $f|_A : A \to Y$ is a submersion. Then $f : (X, A) \to Y$ is a fiber-bundle pair.*

2

Lecture 23 (2013-03-01)

Last time, we were discussing the Ehresmann theorem for fiber bundles of pairs. There was just one thing left to prove.

In the notation of the last lecture, we had C^∞ manifolds X and Y, a closed C^∞ submanifold $A \subseteq X$, and a C^∞ map $f : X \to Y$ such that both f and $f|_A$ are submersions. (Note that for the Ehresmann theorem, we would assume properness, but for now we just want to extract the subbundle W which did not need that hypothesis.)

Proposition. *There exists a subbundle $W \subset TX$ such that*

(i) For all $x \in X$, the derivative $f'(x)|_{W(x)} : W(x) \to T_{f(x)}Y$ is an isomorphism.

(ii) For all $x \in A$, we have $W(x) \subset TA$ (both interpreted as subspaces of T_xX).

This proposition implies the Ehresmann theorem for pairs.

The secret code phrase here is that

$$H^1(\text{any sheaf of modules over the sheaf of } C^\infty \text{ functions}) = 0$$

Proof. For the first step, note that the problem makes sense on any open $U \subset X$, so it will suffice to show that W exists locally, i.e. that for all $x \in X$, there is a neighborhood $U(x)$ where the theorem holds.

If $x \notin A$, then we're done, so suppose that $x \in A$. WLOG, we can take $X = \mathbb{R}^n$, $A = \{x \in \mathbb{R}^n \mid x_{m+1} = \cdots = x_n = 0\}$, and $f : X \to Y$ the map $f(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$ where $r \leq m$. In this case, we can just take W to be the span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}$.

Now to Step 2; we want to provide an algebraic description of W. This is essential. We can't add subbundles, but we can add / do other linear things to sections of bundles.

For each $x \in X$, let $s(x)$ be the inverse of the isomorphism described in (i); in other words, we want to demonstrate the existence of a map of bundles $s : f^*TY \to TX$ such that

(i') $f'(x) \circ s(x) : T_{f(x)}Y \to T_{f(x)}Y$ is the identity for all $x \in X$

(ii') For all $x \in A$, we have $s(x)(T_{f(x)}Y) \subseteq T_xA$.

Step 3: Suppose that s_1 and s_2, both maps $f^*TY \to TX$, satisfy conditions (i') and (ii'). Then $h = s_2 - s_1 : f^*TY \to TX$ satisfies

(i'') $f'(x) \circ h(x) = 0$ for all $x \in X$

(ii'') $h(x)(T_{f(x)}Y) \subseteq T_xA$ for all $x \in A$

so that

$$Z = \{h : f^*TY \to TX \mid \text{(i'') and (ii'') hold}\}$$

is a module over the ring of C^∞ functions on X. Note that this is a characterization; in other words, if s_1 satisfies (i') and (ii'), then $s_1 + h$ satisfies them if and only if $h \in Z$.

As a corollary of Step 3, we see that if $s_1, \ldots, s_m$ are as in Step 2, and $\varphi_1, \ldots, \varphi_m : W \to \mathbb{R}$ are a C^∞ partition of unity (so that $\sum \varphi_i = 1$), then $\sum \varphi_i s_i$ also satisfies the conditions of step 2, because

$$\sum \varphi_i s_i = \underbrace{\sum \varphi_i (s_i - s_1)}_{\in Z} + \underbrace{\left(\sum \varphi_i\right)}_{=1} s_1.$$

Now we come to the proof of the proposition itself. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover equipped with $s_\lambda : f^*TY|_{U_\lambda} \to TX|_{U_\lambda}$ all satisfying (i') and (ii'). There is a partition of unity φ_λ subordinate to U_λ; then $\varphi_\lambda s_\lambda$ (originally defined only on U_λ) can be extended by 0 to a C^∞ map $\varphi_\lambda s_\lambda : f^*TY \to TX$. Now let $s = \sum \varphi_\lambda s_\lambda : f^*TY \to TX$; the corollary above implies that s satisfies (i') and (ii'). ☐

Existence of inner products on vector bundles

Given a C^∞ vector bundle $f : V \to M$, we want to construct a map $B : V \times_M V \to \mathbb{R}$ such that $B : V(x) \times V(x) \to \mathbb{R}$ is a positive definite, symmetric, bilinear form.

If W is a vector space, and $B : W \times W \to \mathbb{R}$ is symmetric and bilinear, we say that B is positive semi-definite if $B(w, w) \geq 0$ for all $w \in W$, and positive definite if it is positive semi-definite and $B(w, w) = 0$ implies $w = 0$.

Proof. Step 1. Assume that $V|_U$ is a trivial bundle, i.e. there exist sections $s_1, \dots, s_k$ of $V|_U$ such that $s_1(x), \dots, s_k(x)$ form a basis for $V(x)$ for all $x \in U$.

Define $B_U(s_i(x), s_j(x)) = \delta_{ij}(x)$. Given an open cover $\mathcal{U}$, and a partition of unity φ_U subordinate to $\mathcal{U}$, then $\sum \varphi_U B_U$ is a symmetric bilinear positive semi-definite form. But for any $x \in X$, if $v \in V(x)$ is non-zero, then there is some U such that $\varphi_U(x) > 0$, so that $x \in U$ and moreover $B_U(v, v) > 0$, hence $B(v, v) \geq \varphi_U(x)B_U(v, v) > 0$. Thus, this is in fact positive definite. ☐

Existence of connections on a vector bundle

A good reference for this is Milnor's *Morse Theory.*

Let $p : V \to M$ be a C^∞ vector bundle. A connection is essentially a way of taking a derivative of a section s of a vector bundle v with respect to a vector field on M.

Suppose that $x \in U$ and that $V|_U$ is trivial, and that $s_1, \dots, s_k$ are sections of $V|_U$ that give a basis for $V(x)$ for each $x \in U$. For any $v \in T_xM$, we define

$$v\left(\sum f_i s_i\right) = \sum v(f_i)s_i.$$

A connection, or a covariant derivative, ∇ on V is a map taking in a vector field v on M, and a section s of V, and outputting $\nabla_v s$, another section of V. We also require that a connection satisfy certain properties: for any C^∞ map $f : M \to \mathbb{R}$,

1. $\nabla_v(s_1 + s_2) = \nabla_v(s_1) + \nabla_v(s_2)$
2. $\nabla_v(fs) = v(f)s + f\nabla_v(s)$ (this is the Leibniz rule)
3. $\nabla_{fv}(s) = f\nabla_v(s)$

We could have stated this definition sheaf-theoretically, which is after all necessary to do it on analytic manifolds, but for C^∞ manifolds, they are equivalent.

We want to show that any C^∞ vector bundle $V \to M$ has a connection.

The argument is the same as we've been doing. Step 1 is to show that they exist locally (this is just the trivial connection). Step 2 is to take two connections ∇^1, ∇^2 and define h via $\nabla^2 = \nabla^1 + h$, i.e. $\nabla^2_v(s) = \nabla^1_v(s) + h_v s$ for all sections s, and note that h satisfies three properties: h is additive in s,

$$h_v(fs) = fh_v(s)$$

for all C^∞ functions f, and $h_{fv}(s) = fh_v(s)$.

Then, if ∇^1 is a connection and $\nabla^2 = \nabla^1 + h$, then ∇^2 is a connection if and only if h satisfies the above three properties. The collection of all such h can be thought of being comprised of precisely the sections of $\mathrm{Hom}(TM, \mathrm{End}(V))$, which is a module over C^∞ functions $M \to \mathbb{R}$.

We then conclude by using a partition of unity and noting that $\sum \varphi_U \nabla_U$ gives a connection.

Let's examine connections in a basic case; let M be an open interval (a, b). By the properties of a connection, all we have to look at is $\nabla_{\frac{d}{dt}}(s)$. In particular, what is

$$\{\text{sections } s : M \to V \mid \nabla_{\frac{d}{dt}}(s) = 0\} \quad ?$$

We know that V is trivial because we're working on an interval; choose a specific trivialization, so that we will think of sections as maps $s : (a, b) \to \mathbb{R}^k$. Define vectors of C^∞ functions m_i by

$$\nabla_{\frac{d}{dt}}(e_i) = m_i,$$

where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

the 1 being in the ith position. Then

Lecture 24 (2013-03-04)

Let $M = (-a,a)$ and consider a vector bundle $V = M \times \mathbb{R}^k$ on M. If s is a section of V, i.e. a map $s : (-a,a) \to \mathbb{R}^k$, then $\nabla_{\frac{d}{dt}}(s) = \frac{ds}{dt} + Gs$ where G.

We want to solve $\nabla_{\frac{d}{dt}} s_i = 0$, where $s_i(0) = e_i$. If this is defined for all $t \in (-a_i, a_i)$, let $\epsilon = \min\{a_i\}$. It follows that for all $w \in \mathbb{R}^k$, $w = \sum_{i=1}^k w_i e_i$, we have that $\nabla_{\frac{d}{dt}}(\sum w_i s_i) = 0$ and $(\sum w_i s_i)(0) = w$. By the compactness of an interval $[p,q]$, it follows that for all $w \in V(p)$ there is a unique section s of V on $[p,q]$ such that $\nabla_{\frac{d}{dt}}(s) = 0$ and $s(p) = w$.

This yields the notion of parallel translation. Suppose we are given M, a vector bundle V on M equipped with a connection ∇, and a C^∞ path $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Given a $w \in V(p)$, we then get $P(\gamma) : V(p) \xrightarrow{\cong} V(q)$, by considering the unique section s

$$\begin{tikzcd} & V \arrow[d] \\ {[0,1]} \arrow[ur, "s"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

with $\nabla_{\frac{d}{dt}}(s) = 0$ and $s(0) = w$. We then put $P(\gamma)(w) = s(a)$.

Note that if $\gamma_1, \gamma_2 : [0,1] \to M$ are paths from p to q that are homotopic, then it need not be the case that $P(\gamma_1) = P(\gamma_2)$.

Take a look at the section on connections on principal bundles in Kobayashi and Nomizu.

Given a vector bundle $\pi : V \to M$ equipped with a connection ∇, we get a subbundle H of TV such that $\pi'(\xi) : H_\xi \xrightarrow{\cong} T_{\pi(\xi)}M$.

Non-canonically, we can get it like this. Let $M =$ open $\omega \subset \mathbb{R}^n$, and let V be a trivial bundle on M of rank k. Define maps G_i by $\nabla_{\partial_i}(e_j) = G_i(e_j)$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$. The maps G_i are really maps from Ω to $\mathrm{M}_k(\mathbb{R})$. The subbundle $H = H(\nabla)$ of $T(\Omega \times \mathbb{R}^k)$ is the linear span of the sections θ_i, given by $(x,s) \mapsto (e_i, -G_i(x)s)$, for $i = 1, \ldots, n$. In particular, θ_i is a vector field on $\Omega \times \mathbb{R}^k$. The integral curves of the θ_i are of the type $t \mapsto (x + te_i, s(t)$ where $\nabla_{\partial_i}(s) = 0$.

Definition. Given a vector bundle V and connection ∇, a section s

$$\begin{tikzcd} & V \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

we say that s is horizontal if $\nabla_X(s) = 0$ for all vector fields X on U.

Definition. A connection ∇ is trivial if there exists horizontal sections $s_1, \ldots, s_k$ such that $s_1(x), \ldots, s_k(x)$ is a basis for $V(x)$ for all $x \in M$.

Definition. A vector bundle V with connection ∇ is integrable (a.k.a. flat) if M has an open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ such that $(V, \nabla)|_{U_\lambda}$ is trivial.

Our previous results about the Frobenius theorem demonstrate that

$$(V, \nabla) \text{ is integrable} \iff H(\nabla) \text{ is a trivial subbundle} \iff [\theta_i, \theta_j] = 0 \text{ for all } 1 \le i,j \le n$$

$$\iff [\nabla_{\frac{\partial}{\partial x_i}}, \nabla_{\frac{\partial}{\partial x_j}}] = 0 \text{ for all } 1 \leq i, j \leq n$$

Now the issue is, how can we put these criteria in a usable form?

Remark. Consider a map L of vector bundles

$$\begin{tikzcd} V_1 \arrow[rr, "L"] \arrow[dr, "\pi_1"'] & & V_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

Then for any open $U \subseteq M$, L induces a map $C^\infty(U, V_1) \to C^\infty(U, V_2)$, so that L is a homomorphism of sheaves of C^∞_M-modules, i.e. for any $f : U \to \mathbb{R}$ and $s : U \to V_1$, we have that $L(fs) = fL(s)$. The remark is that the converse is true: given vector bundles V_1, V_2 and $T : C^\infty(M, V_1) \to C^\infty(M, V_2)$ with $T(fs) = fT(s)$ for all $f : M \to \mathbb{R}$, then there is a unique bundle homomorphism $L : V_1 \to V_2$ such that $T(s) = L \circ s$ for all sections s.

Example. Let S be a subbundle of TM and let $Q = TM/S$ be the quotient bundle. Let $p : TM \to Q$ be the projection map. Let s_1, s_2 be sections of S. Then $[s_1, s_2]$ is a section of S if and only if $p([s_1, s_2]) = 0$. Define the map $B : C^\infty(M; S) \times C^\infty(M; S) \to C^\infty(M; Q)$ by $B(s_1, s_2) \mapsto p([s_1, s_2])$.

As a lemma, we claim that $B(fs_1, s_2) = fB(s_1, s_2)$ for all $f : M \to \mathbb{R}$, and $B(s_1, s_2) = -B(s_2, s_1)$. This follows from the fact that

$$[fv_1, v_2] = -[v_2, fv_1] = -v_2(f)v_1 - f[v_2, v_1] = f[v_1, v_2] - v_2(f)v_1,$$

and that $v_2(f)v_1$ is a section of S so that p of it is 0.

Definition. Let (V, ∇) be a vector bundle with connection on M. Define

$$R(X, Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}(s),$$

where s is a section of V, and X and Y are sections of TM. The lemma we proved above implies that

$$R(f_1 X, f_2 Y)(f_3 s) = f_1 f_2 f_3 R(X, Y)(s)$$

for all $f_i : M \to \mathbb{R}$ and X, Y.

Corollary. *The function R defines a homomorphism $R : \Lambda^2 TM \to \mathrm{End}(V)$.*

When Ω is an open subset of $\mathbb{R}^n$,

$$R = 0 \iff R(\partial_i, \partial_j) = 0 \text{ for all } i, j \iff \nabla_{\partial_i} \nabla_{\partial_j} = \nabla_{\partial_j} \nabla_{\partial_i} \text{ for all } i, j.$$

Grassmannians

Let V be a fixed vector space over $\mathbb{R}$. We define $M = \mathrm{Grass}(r, V)$ to be the collection of rank-r subspaces of V. We can define the tautological bundle on M to be the subbundle S of $M \times V$ to be

$$\{(W, v) \mid W \in \mathrm{Grass}(r, V), v \in W\}.$$

Define $Q = (M \times V)/S$, and let $p : M \times V \to Q$. The bundle $M \times V$ has the trivial connection ∇, so $\nabla(v) = 0$ for all constant sections v.

Lecture 25 (2013-03-06)

Lie Groups

A Lie group G is a C^∞ manifold G equipped with a C^∞ binary operation $B : G \times G \to G$ that turns the set G into a group. As usual, we just write xy for $B(x, y)$.

For any $g \in G$, there is a C^∞ map $\ell_g : G \to G$ defined by $\ell_g(x) = gx$ for all $x \in G$. We can see that this is C^∞ because it is the composition

$$\begin{tikzcd} G \arrow[r] & G \times G \arrow[r, "B"] & G \\ x \arrow[r, mapsto] & (g, x) \arrow[r, mapsto] & gx \end{tikzcd}$$

Clearly, $\ell_g \circ \ell_{g^{-1}} = \mathrm{id}_G$ for all $g \in G$. Thus $\ell_g : G \to G$ is a diffeomorphism.

Let $A : G \times G \to G \times G$ be defined by $A(x, y) = (x, xy) = (x, B(x, y))$. We then have a commutative diagram

$$\begin{tikzcd} G \times G \arrow[r, "A"] \arrow[d, "p_1"'] & G \times G \arrow[d, "p_1"] \\ G \arrow[r, "\mathrm{id}"'] & G \end{tikzcd}$$

Clearly A is a bijection, and its inverse is F, defined by $F(x, y) = (x, x^{-1}y)$. Observe that A is C^∞, and furthermore that A induces isomorphisms on all tangent spaces; we can see the latter as follows: identifying $T_{(x,y)}(G \times G) \cong T_xG \oplus T_yG$, we have

$$A'(x, y)(0, a) = (0, \ell'_x(y)a),$$

so that $A'(x, y)$ maps $0 \oplus T_yG$ isomorphically to $0 \oplus T_{xy}G$, and similarly

$$A'(x, y)(a, 0) = (a, \text{something})$$

which already suffices to show that $A'(x, y)$ is an isomorphism. It follows from these observations that A is a diffeomorphism, so that its inverse F is C^∞. In particular, setting $y = e$, we see that $x \mapsto (x, e) \mapsto F(x, e) = (x, x^{-1})$ is a C^∞ function. Thus $x \mapsto x^{-1}$ is a C^∞ fucntion from G to G.

Definition. The Lie algebra of G is defined to be

$$\mathrm{Lie}(G) = \{\text{vector fields } v \text{ on } G \mid \ell_g^* v = v \text{ for all } g \in G\}.$$

In other words, it is the collection of left-invariant vector fields on G.

Observe that $v \mapsto v(e)$ gives an isomorphism $I_G : \mathrm{Lie}(G) \to T_eG$ of real vector spaces. We can see this as follows. For any $v \in \mathrm{Lie}(G)$, left-invariance implies that $v(g) = \ell'_g(e)v(e)$ for all $g \in G$, where $v(e) \in T_eG$. This shows that I is injective.

To show that I is surjective, we note that for any $a \in T_eG$, setting $v(g) = \ell'_g(e)a$ defines a C^∞ vector field v on G.

Proposition. *If $v, w \in \mathrm{Lie}(G)$, then $[v, w] \in \mathrm{Lie}(G)$.*

Proof. This is straightforward. Because Lie bracket respects diffeomorphisms,

$$[v, w] = [\ell_g^* v, \ell_g^* w] = \ell_g^* [v, w]$$

for all $g \in G$. ☐

Recall that for a C^∞ map $f : M \to N$, vector fields X and Y on M and N, respectively, are said to be f-related when

$$f^* X(x) = Y(f(x))$$

for all $x \in M$.

Proposition. *Let $f : M \to N$ be a C^∞ map.*

1. *γ is an integral curve of $X \implies f \circ \gamma$ is an integral curve of Y*
2. *If X and X' are f-related to Y and Y' respectively, then $[X, X']$ is f-related to $[Y, Y']$.*

We say that $f : H \to G$ is a homomorphism of Lie groups when f is C^∞ and also a group homomorphism. There is a corresponding map df obtained as the composition

$$\begin{tikzcd} \mathrm{Lie}(H) \arrow[r, "I_H", "\cong"'] \arrow[rrr, bend right, "df"'] & T_eH \arrow[r, "f'(e)"] & T_eG \arrow[r, "I_G^{-1}", "\cong"'] & \mathrm{Lie}(G) \end{tikzcd}$$

Lemma. *For all $X \in \mathrm{Lie}(H)$, we have*

1. *X is f-related to $df(X)$.*
2. *$df : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ is a homomorphism of Lie algebras.*

Proof. For all $h \in H$, we have a commutative diagram

$$\begin{tikzcd} H \arrow[r, "\ell_h"] \arrow[d, "f"'] & H \arrow[d, "f"] \\ G \arrow[r, "\ell_{f(h)}"'] & G \end{tikzcd}$$

which proves claim 1. To see claim 2, just note that df is $\mathbb{R}$-linear and

$$df[X_1, X_2] = [df(X_1), df(X_2)]$$

by part 2 of the earlier proposition. ☐

Let $X \in \mathrm{Lie}(G)$. Let $\gamma : (-\epsilon, \epsilon) \to G$ be an integral curve of X. The fact that $\ell_g^* X = X$ implies that $\ell_g \circ \gamma : (-\epsilon, \epsilon) \to G$ is also an integral curve of X. We know that an integral curve $\gamma_e : (-\epsilon, \epsilon) \to G$ can be defined for some $\epsilon > 0$, so that $\gamma_g : (-\epsilon, \epsilon) \to G$ given by $\gamma_g(t) = g\gamma_e(t)$ is the unique integral curve of X with $\gamma_g(0) = g$.

Thus, $(x, t) \mapsto \gamma_x(t) = \phi_t(x)$ is defined on $G \times (-\epsilon, \epsilon)$. This implies that $(x, t) \mapsto \phi_t(x)$ is defined on all of $G \times \mathbb{R}$. Thus, we see that

- $\phi_t(x) = x\phi_t(e)$ for all $x \in G$ and $t \in \mathbb{R}$.
- $\phi_t \circ \phi_s = \phi_{t+s}$ for all t, s, which implies that $\phi_t(e)\phi_s(e) = \phi_{t+s}(e)$ for all t, s, or in other words, $\gamma_e(t)\gamma_e(s) = \gamma_e(t + s)$ for all t, s.

Corollary. *For all $X \in \mathrm{Lie}(G)$, there is a unique homomorphism of Lie groups $f : \mathbb{R} \to G$ such that $f'(0)\frac{d}{dt} = X(0)$, or equivalently, that $\frac{d}{dt}$ is f-related to X.*

Proof. To see existence, just take $f = \gamma_e$. For uniqueness, note that if we have a homomorphism $f : \mathbb{R} \to G$ such that $f'(0)\frac{d}{dt} = X(0)$, then $\frac{d}{dt}$ and x are f-related, so that $t \mapsto f(t)$ is an integral curve of f by part 1 of our earlier proposition. The uniqueness of integral curves of a vector field through a point gives uniqueness of f. ☐

We will denote the above map $f : \mathbb{R} \to G$ be f_X, for each $X \in \mathrm{Lie}(G)$.

Definition. We define the exponential map $\exp : \mathrm{Lie}(G) \to G$ by $\exp(X) = f_X(1)$. This is a C^∞ map, because it varies smoothly on its parameters, and also

$$\exp'(0) : \mathrm{Lie}(G) \to T_gG \xrightarrow{I_G^{-1}} \mathrm{Lie}(G)$$

is the identity map.

Observe that if v is a vector field on M and γ is an integral curve of v, then for all $c \in \mathbb{R}$, the map $t \mapsto \gamma(ct)$ is of course an integral curve of cv. Thus, $f_X(t) = \exp(tX)$ for all $t \in \mathbb{R}$ and $X \in \mathrm{Lie}(G)$. This shows that $\exp'(0) = \mathrm{id}_{\mathrm{Lie}(G)}$.

We've seen that a Lie group gives rise to a Lie algebra. In general you can't go back from Lie algebras to Lie groups, because you have things like covering spaces for example sharing the same Lie algebra. However, there is a correspondence between simply-connected Lie groups and finite-dimensional $\mathbb{R}$-Lie algebras.

Let M be a C^∞ manifold and let S be a subbundle of TM. Let's say that S is involutive, which is also sometimes called being a foliation. We give M a new topology, which we'll denote M^{new}, which has as a basis

$$\{N \subseteq M \mid N \text{ is a locally closed submanifold and } T_xN = S(x) \text{ for all } x \in N\}.$$

The Frobenius theorem implies that the requirements of a basis are met.

Definition. A leaf of S is a connected component of M^{new} (this is obviously a C^∞ manifold of dimension r where $r = \mathrm{rank}(S)$).

Lecture 26 (2013-03-08)

Let us start with the following data: M is a C^∞ manifold, $S \subset TM$ is an involutive subbundle, and

$$B = \{N \subseteq M \mid N \text{ is a locally closed submanifold of } M, \text{ and } T_xN = S(x) \text{ for all } x \in N\}.$$

The set B forms a basis for a topology on M, which we'll denote M^{new}. Then M^{new} is a C^∞ manifold. A leaf of S is defined to be a connected component of M^{new}.

Exercise. If M is second-countable and Hausdorff, then every leaf is second-countable. Note that because $M^{\text{new}} \to M$ is continuous and M is Hausdorff, we have that M^{new} is Hausdorff.

Here is an application of this.

Let G be a Lie group, and let $W \subset \text{Lie}(G)$ be a Lie subalgebra (i.e. W is closed under taking Lie brackets). Let $\langle W \rangle$ be the subbundle spanned by W, so that if $X_1, \ldots, X_k$ are an $\mathbb{R}$-basis of W, then $\langle W \rangle(g)$ is the span of $X_1(g), \ldots, X_k(g)$ for all $g \in G$. Note that $\langle W \rangle$ is involutive; this is because any section of W is of the form $\sum_{i=1}^{i} f_i X_i$, and just using the rule $[X, fY] = X(f)Y + f[X, Y]$.

Let H be the connected component of G^{new} containing the identity $e \in G$. From the definition of G^{new} (which, recall, depends on the W we initially chose), it is clear that the tangent space of H at e is

$$T_eH = \{X(e) \in T_eG \mid X \in W\}.$$

Because $\ell_g^* X = X$ for all $X \in \text{Lie}(G)$ and $g \in G$, we see that for any linear subspace $W \subset \text{Lie}(G)$, we have $\ell_g^* \langle W \rangle = \langle W \rangle$. It follows that the leaves are permuted amongst each other, and thus $\ell_g : G^{\text{new}} \to G^{\text{new}}$ is continuous. Thus, for any $a \in H$, it follows that $a^{-1}H$ is also a leaf; but $e \in a^{-1}H$, and $a^{-1}H$ is a connected component, so we must have $H = a^{-1}H$. Thus H is a subgroup.

Corollary. *Let W and G be as above. Then there is a Lie group H and Lie group homomorphism $f : H \to G$ such that*

$$\begin{tikzcd}
\text{Lie}(H) \arrow[rr, "df"] \arrow[dr, "\cong"'] & & \text{Lie}(G) \\
& W \arrow[ur, hook] &
\end{tikzcd}$$

Lemma. *Let G be a connected Lie group. Let $p : \widetilde{G} \to G$ be a universal covering space. Let $\widetilde{e} \in G$ be an element of $p^{-1}(e)$. Then $\widetilde{G}$ with its natural C^∞ structure has the structure of a Lie group with $\widetilde{e}$ as its identity and $p : \widetilde{G} \to G$ a homomorphism of Lie groups.*

Proof. $\widetilde{G} \times \widetilde{G}$ is simply connected and p is a covering map, so by covering space theory, in the following diagram

$$\begin{tikzcd}
\widetilde{G} \times \widetilde{G} \arrow[d, "p \times p"'] & G \arrow[d, "p"] \\
G \times G \arrow[r, "B"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
(\widetilde{e}, \widetilde{e}) \arrow[d, mapsto] & \widetilde{e} \arrow[d, mapsto] \\
(e, e) \arrow[r, mapsto] & e
\end{tikzcd}$$

there is a unique lift $\widetilde{B} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $\widetilde{B}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$ and $p \circ \widetilde{B} = B \circ (p \times p)$. The binary operation that B defines on G will just be written as juxtaposition.

We can define maps P and Q

$$\begin{tikzcd}
\widetilde{G}\times\widetilde{G}\times\widetilde{G} \arrow[r, shift left, bend left=5, "P"] \arrow[r, shift right, bend right=5, "Q"'] \arrow[dr] & \widetilde{G} \arrow[d, "p"] \\
 & G
\end{tikzcd}$$

by $P(a,b,c) = a(bc)$ and $Q(a,b,c) = (ab)c$, and the diagonal is just $p(a)p(b)p(c)$. It follows taht there is a covering transformation $\gamma : \widetilde{G} \to \widetilde{G}$ of p such that $Q = \gamma \circ P$. But $P(\widetilde{e},\widetilde{e},\widetilde{e}) = Q(\widetilde{e},\widetilde{e},\widetilde{e})$. The rest of the arguments are similar and are omitted. $\square$

Theorem. *The functor* $G \mapsto \mathrm{Lie}(G)$ *from the category of simply connected Lie groups and Lie group homomorphisms to the category of Lie algebras and Lie algebra homomorphisms is an equivalence of categories.*

In particular, given Lie groups G_1 and G_2, then $\mathrm{Mor}(G_1, G_2) \to \mathrm{Mor}(\mathrm{Lie}(G_1), \mathrm{Lie}(G_2))$ is a bijection. To see that it is injective, note that we know this is true when $G_1 = \mathbb{R}$, i.e. that $\mathrm{Mor}(\mathbb{R}, G) \to \mathrm{Lie}(G) = \mathrm{Mor}(\mathbb{R}, \mathrm{Lie}(G))$ is injective, so when given $f, g : G_1 \to G_2$ such that $df = dg : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$, let $X \in \mathrm{Lie}(G_1)$; the maps $\gamma : t \mapsto f(\exp(tX))$ and $\delta : t \mapsto g(\exp(tX))$ satisfy

$$\gamma'(0) = df(X)(e) = dg(X)(e) = \delta'(0),$$

so that $f(\exp(tX)) = g(\exp(tX))$ for all $X \in \mathrm{Lie}(G_1)$ and $t \in \mathbb{R}$.

Now note that $M = \{z \in G_1 \mid f(z) = g(z)\}$ is clearly a subgroup of G_1. The set $\{\exp(X) \mid X \in \mathrm{Lie}(G)\}$ contains a neighborhood U of $e \in G$. Because G is connected, we therefore must have that $M = G$. Thus $f = g$.

Now we prove that the map $\mathrm{Mor}(G_1, G_2) \to \mathrm{Mor}(\mathrm{Lie}(G_1), \mathrm{Lie}(G_2))$ is surjective.

Given a homomorphism $L : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ of Lie algebras, we want to show it comes from a Lie group homomorphism. Let $G = G_1 \times G_2$, so that $\mathrm{Lie}(G) = \mathrm{Lie}(G_1) \times \mathrm{Lie}(G_2)$. If p_1, p_2 are the projections, then for any vector fields v_1, v_2 on G_1, G_2, we have that $p_i^* v_i$ is a section of $p_i^* TG_i$, and there is a natural inclusion $p_i^* TG_i \hookrightarrow TG$, so we may regard each $p_i^* v_i$ as a section of TG.

We claim that $[p_1^* v_1, p_2^* v_2] = 0$. Let $W = \{(\alpha, L(\alpha)) \mid \alpha \in \mathrm{Lie}(G)\} \subset \mathrm{Lie}(G)$. Because L is a Lie algebra homomorphism, W is a Lie subalgebra. It follows that there is a connected H and map $\rho : H \to G$ such that the image of $d\rho : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ is W. We claim that the map σ, in the diagram

$$\begin{tikzcd}
H \arrow[r, "\rho"] \arrow[dr, "\sigma"'] & G_1 \times G_2 \arrow[d, "p_1"] \\
 & G_1
\end{tikzcd}$$

is an isomorphism. Note that, if this is so, then $p_2 \circ \rho \circ \sigma^{-1}$ is the desired map $G_1 \to G_2$.

Let's prove the claim. We know that $d\sigma : \mathrm{Lie}(H) \to \mathrm{Lie}(G_1)$ is an isomorphism, so it suffices to show that σ is a covering map (because G_1 is simply connected). We make the following observations:

1. The image $\mathrm{im}(\sigma)$ is a subgroup, $\mathrm{im}(\sigma)$ contains a non-empty open subset, and G_1 connected $\implies \mathrm{im}(\sigma) = G_1$
2. The map $\sigma : H \to G_1$ is a group homomorphism and surjective. It now suffices to check that there is a neighborhood U of $e \in G_1$ that is evenly covered.

We have a neighborhood P of $e \in H$ such that $\sigma|_P : P \to \sigma(P)$ is a homeomorphism. By continuity, there is a neighborhood Q of e such that $Q \cdot Q \subset P$. Assume that $Q = Q^{-1}$ in addition. Let $\Gamma = \ker(\sigma)$; we claim that $\Gamma \times Q \to \sigma^{-1}(\sigma(Q))$, $(\gamma, q) \mapsto \gamma q$ is a homeomorphism.

Next time we'll finish this proof.

Lecture 27 (2013-03-11)

Lecture 28 (2013-03-13)

Let G be a Lie group. A tangent vector at the identity can be extended uniquely to a left-invariant vector field on G, i.e. an element of $\mathrm{Lie}(G)$. Similarly, it can be extended in a unique way to a right-invariant vector field on G. Let us call the collection of these $\mathrm{Lie}^r(G)$. If $i : G \to G$ is the inverse map $i(g) = g^{-1}$, then we clearly see that $X \in \mathrm{Lie}(G)$ if and only if $i^*X \in \mathrm{Lie}^r(G)$. Because i induces -1 on the tangent space T_eG, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Lie}(G) \arrow[r, "i^*"] \arrow[d, "\mathrm{ev}_g"'] & \mathrm{Lie}^r(G) \arrow[d, "\mathrm{ev}_g"] \\
T_eG \arrow[r, "-1"'] & T_eG
\end{tikzcd}$$

where ev_e is evaluation at the identity (we've previously called this map I_G).

Given a C^∞ homomorphism $\sigma : G \to \mathrm{Diffeo}(M)$, we get a C^∞ group action $G \times M \to M$ given by $(g, m) \mapsto \sigma(g)m$.

Let $m \in M$. Define $f_m : G \to M$ by $f_m(g) = \sigma(g)m$. We have the derivative $f'_m(e) : T_eG \to T_mM$. For all $v \in T_eG$, we get a vector field $\widetilde{v}$ on M defined by $\widetilde{m} = f'_m(e)v$. This is a map from T_eG to C^∞ vector fields on M.

For all $v \in T_eG$, let $[v]$ be the unique right-invariant vector field with $[v](e) = v$. Then $[v](g) = r'_g(e)v$ for all $g \in G$, where r_g is right multiplication by g.

Lemma.

1. *Let $m \in M$. Let $v \in T_eG$. Then for $f_m : G \to M$, $[v]$ is f_m-related to $\widetilde{v}$.*
2. *If $v, w \in T_eG$, then $[[v], [w]]$ is f_m-related to $[\widetilde{v}, \widetilde{w}]$.*
3. *To rephrase the above, the map $\mathrm{Lie}^r(G) \to$ vector fields on M defined by $[v] \mapsto \widetilde{v}$ is a Lie algebra homomorphism.*

Proof. Let's prove 1 first. Let $g \in G$. We have $\widetilde{v}(gm) = f'_{gm}(e)v = f'_m(g)r'_g(e)v$ (we are now neglecting to write the σ). But also

$$f_{gm}(g) = hgm = f_m(hg) = (f_m \circ r_g)(h) = f'_m(g)[v](g),$$

which is the same. ☐

Proposition (Converse)**.** *There is also a converse: a Lie algebra homomorphism $h : \mathrm{Lie}^r(G) \to C^\infty$ vector fields on M yields an action of G on M, if we assume that (a) G is simply connected and (b) M is compact.*

Proof. Each $w \in \mathrm{Lie}^r(G)$ produces a vector field $\theta(w)$ on $G \times M$ defined by

$$\theta(w)(g, m) = (w(g), h(w)(m)).$$

We claim that $w \mapsto \theta(w)$ is a homomorphism of Lie algebras. Note that the map $\mathrm{Vect}(M) \times \mathrm{Vect}(N) \to \mathrm{Vect}(M \times N)$ defined by $(v, w) \mapsto (p_1^*v, p_2^*w)$ is a Lie algebra homomorphism. Thus, we have an involutive subbundle $S(g, m) = \{\theta(w)(g, m) \mid w \in \mathrm{Lie}^r(G)\}$ (observation 1). Observe (2) that

$$\begin{tikzcd} S(g,m) \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & T_{(g,m)}(G\times M) \arrow[r] & T_gG \end{tikzcd}$$

It follows that a leaf is, locally, the graph of a function $U \to M$ for some $U \hookrightarrow G$. We get the following observation (3): for all $m \in M$, there is a neighborhood U of $e \in G$, a neighborhood U_m of $m \in M$, and a map $f : U \times U_m \to M$ such that for all $m' \in U_m$, the image of $g \mapsto (g, f(g, m'))$ has tangent space S, and $f(e, m') = m'$ for all $m' \in U_m$.

Observation 4: Assuming that M is compact, we get a neighborhood U of $e \in G$ and $f : U \times M \to M$ with $f(e, m) = m$ for all $m \in M$, and a leaf $g \mapsto (g, f(g, m))$ of S, for all $m \in M$.

Observation 5: Giving $G \times M$ the topology it gets as a disjoint union of leaves, we have a commutative diagram

$$\begin{tikzcd} (U\times M)^{\text{new}} \arrow[r, hook, "\text{open}"] \arrow[d, "\substack{\text{covering}\\\text{map}}"'] & (G\times M)^{\text{new}} \arrow[d] \\ U \arrow[r] & G \end{tikzcd}$$

where we know this map is a covering map by 4.

□

The adjoint representation

Let G be a Lie group. For any $g \in G$, there is the conjugation map $C_g : G \to G$ defined by $C_g(h) = ghg^{-1}$. Clearly $C_{g_1} \circ C_{g_2} = C_{g_1 g_2}$. The derivative $C'_g(e) : T_eG \to T_eG$ is denoted by $\text{Ad}(g) : T_eG \to T_eG$. We have that Ad is an action of G on $\text{Lie}(G)$. As an exercise, compute $d(\text{Ad})$, which is an action of $\text{Lie}(G)$ on itself.

Let G be a Lie group and H a closed Lie subgroup. We say that $\pi : G \to G/H$ is a homogeneous manifold. A homogeneous vector bundle on G/H is just a vector bundle W on G equipped with an action of G such that the diagram

$$\begin{tikzcd} W \arrow[r, "\sigma(g)"] \arrow[d, "p"'] & W \arrow[d, "p"] \\ G/H \arrow[r, "\ell_g"'] & G/H \end{tikzcd}$$

commutes, and such that $\sigma(g) : W(x) \to W(x)$ is a linear transformation for all $g \in G$ and $x \in G$.

Observe that $\ell_h \overline{e} = \overline{e}$ for all $h \in H$, where $\overline{e} = eH \in G/H$. It follows that $\sigma(h)(W_{\overline{e}}) = W_{\overline{e}}$. We see that $h \mapsto \sigma(h)|_{W_{\overline{e}}}$ is a representation on H on $W(\overline{e})$. Conversely, given M and a representation of H, we can construct W as the quotient of $G \times M$ by $(gh, m) \sim (g, hm)$ for all $g \in G$, $h \in H$, $m \in M$.

Lecture 29 (2013-03-15)

This lecture is really an apology for not having done any examples.

Classical groups over $\mathbb{R}$ and $\mathbb{C}$, and their maximal compact subgroups

There's a paper by Andre Weil, from around 1955, titled *Algebras with Involution*, which says what we'll do today but over all fields. Another good reference is Helgason's *Differential Geometry and Symmetric Spaces*.

Let V be a finite-dimensional vector space over K, where $K = \mathbb{R}$, $K = \mathbb{C}$, or $K = \mathbb{H}$ (the quaternions). In each case, we have an involution $K \to K$ denoted by $z \mapsto \overline{z}$, which is additive and reverses multiplication, i.e.

$$\overline{zw} = \overline{w}\,\overline{z}.$$

Given a (left) finite-dimensional vector space V over K, we say that $B : V \times V \to K$ is sesquilinear if it is additive in each variable separately, and

$$B(\lambda v, \mu w) = \lambda B(v, w)\overline{\mu}$$

for all $\lambda, \mu \in K$ and $v, w \in V$.

We say that a sesquilinear form B is positive definite when for all non-zero $v \in V$, we have that $B(v, v) \in \mathbb{R}$, and that $B(v, v) > 0$.

This B is essentially unique, as we will now see. If $e_1, \dots, e_n$ is a K-basis for V, then a standard example of such a B is

$$B\left(\sum \lambda_i e_i, \sum \mu_j e_j\right) = \sum_{i=1}^{n} \lambda_i \overline{\mu}_i.$$

An easy exercise is that $\mathrm{GL}_K(V)$ acts transitively on the space of sesquilinear, positive definite forms $B : V \times V \to K$.

We won't prove this, but any compact group G has a left-invariant measure which is unique up to scaling. Thus, there is a unique left-invariant measure μ such that $\mu(G) = 1$. This lets us treat G as if it were finite.

Corollary 1. *If $\rho : G \to \mathrm{GL}(V)$ is a representation of a compact group G, then for all $v \in V$, we can define*

$$Pv = \int_G \rho(g)v \, d\mu(g),$$

and the map P is a projection of V onto the subspace

$$V^G = \{v \in V \mid \rho(g)v = v \text{ for all } g \in G\}.$$

In other words, $Pv \in V^G$ for all $v \in V$, and if $v \in V^G$, then $Pv = v$.

Corollary 2. *If $\rho : G \to \mathrm{GL}(V)$ is a representation of a compact group G, then there is a unique positive definite sesquilinear form $B : V \times V \to K$ such that*

$$B(\rho(g)v, \rho(g)w) = B(v, w)$$

for all $g \in G$ and $v, w \in V$.

Proof. Let B be any positive definite sesquilinear form on V. Define $\widetilde{B}$ by

$$\widetilde{B}(v, w) = \int_G B(\rho(g)v, \rho(g)w)\, d\mu(g).$$

It is clear that $\widetilde{B}$ has the desired properties. ☐

Let A be a finite-dimensional associative algebra over $\mathbb{R}$. The units of A, denoted $A^\times$, form an open subset of A. $A^\times$ is a Lie group, with Lie algebra $\mathrm{Lie}(A^\times) = A$, and for all $X, Y \in \mathrm{Lie}(A^\times)$, we have $[X, Y] = XY - YX$.

Now assume that A is simple, which forces that $A = \mathrm{End}_K(V)$ where V is some finite-dimensional K-vector space, where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We have that $A^\times = \mathrm{GL}_K(V)$.

Observe that, given a positive definite, sesquilinear form B on V, the set

$$M = \{g \in \mathrm{GL}_K(V) \mid B(gv, gw) = B(v, w) \text{ for all } v, w \in V\}.$$

is a compact group; indeed, every maximal compact subgroup of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$ is a conjugate of $\mathrm{O}_n(\mathbb{R})$, $\mathrm{U}_n(\mathbb{C})$, or $\mathrm{Sp}_n(\mathbb{H})$ respectively.

Theorem. *Let G be a Lie group with finitely many connected components. Then there exist maximal compact subgroups of G, and they are all conjugate to each other.*

Definition. Given an associative ring A, an involution on A is a map $i : A \to A$ that is additive, $i(ab) = i(b)i(a)$ for all $a, b \in A$, and $i(i(a)) = a$ for all $a \in A$.

Let

$$G = \{a \in A \mid ai(a) = 1 = i(a)a\},$$

i.e. the subset consisting of $a \in A$ such that $i(a) = a^{-1}$. Note that

$$i(ab) = i(b)i(a) = b^{-1}a^{-1} = (ab)^{-1} = i(ab),$$

so that G is a subgroup. As an exercise, you can show using the implicit function theorem on the map $A \to A^+ = \{a \in A \mid a = i(a)\}$ defined by $a \mapsto ai(a)$, that G is a Lie group and that $\mathrm{Lie}(G) = A^-$ (A^+ and A^- are the eigenspaces of i for $+1$ and -1, respectively).

Let $B : V \times V \to \mathbb{R}$ be a non-degenerate bilinear form. When we require that B is symmetric, we write $\mathrm{O}(B)$ for the stabilizer of B in $\mathrm{GL}_\mathbb{R}(V)$. When we require that B is antisymmetric, we write $\mathrm{Sp}(B)$ for the stabilizer of B in $\mathrm{GL}_\mathbb{R}(V)$.

Let's suppose that B is symmetric for now. Then there is a basis $e_1, \ldots, e_n$ of V such that $B(e_i, e_j) = 0$ for $i \neq j$, and for some $p + q = n$, we have

$$B(e_i, e_i) = \begin{cases} 1 & \text{if } i = 1, \ldots, p, \\ -1 & \text{if } i = p+1, \ldots, n. \end{cases}$$

By Sylvester's law of inertia, p depends only on B. We write $\mathrm{O}(B) = \mathrm{O}(p, q)$.

Let H be a compact subgroup of $\mathrm{O}(B)$, where B is a non-degenerate symmetric bilinear form on V. We know that H preserves a positive definite form $\langle\,\cdot\,,\,\cdot\,\rangle$. It follows that the map $T : V \to V$ defined by $B(x, y) = \langle Tx, y\rangle$ for all $x, y \in V$ satisfies $T = T^*$, and because H preserves B, we have that T commutes with the H action. Because B is non-degenerate, we have that T is invertible.

Let $V = \bigoplus_{\lambda \in \mathbb{R}} V_\lambda$ be the eigenspace decomposition of $T : V \to V$ (note that the V_λ's are $\langle \cdot , \cdot \rangle$ orthogonal). We have that $H(V_\lambda) \subseteq V_\lambda$ for all λ.

Let $V^+ = \bigoplus_{\lambda > 0} V_\lambda$ and $V^- = \bigoplus_{\lambda < 0} V_\lambda$, so that $V = V^+ \oplus V^-$. We see that $B|_{V^+}$ is positive definite, and $-B|_{V^-}$ is positive definite, and $B(v, w) = 0$ for all $v \in V^+$ and $w \in V^-$.

Thus, we have

$$\underbrace{H}_{\text{compact}} \hookrightarrow \mathrm{O}(V^+, B|_{V^+}) \times \mathrm{O}(V^-, B|_{V^-}) \hookrightarrow \mathrm{O}(V, B)$$

For any $W \subseteq V$ such that $B|_W$ is positive definite and $B|_{W^\perp}$ is negative definite, we have that

$$\{g \in \mathrm{O}(V, B) \mid gW = W\}$$

is a maximal compact subgroup. Thus, maximal compact subgroups are in bijection with

$$\{W \subseteq V \mid B|_W \text{ is positive definite, and } B|_{W^\perp} \text{ is negative definite}\}.$$

This completes the symmetric case. Now suppose that B is skew-symmetric, and let $G = \mathrm{Sp}(V, B)$. Let $V = W$, where W is a k-dimensional $\mathbb{C}$-vector space, and $n = 2k$. Let $H : W \times W \to \mathbb{C}$ be a positive definite Hermitian form, so that $\mathrm{U}(H) = \mathrm{U}(k)$ is a compact subgroup of $\mathrm{GL}_{\mathbb{C}}(W)$. If $g \in \mathrm{U}(H)$, then $\mathrm{Im}(H(gv, gw)) = \mathrm{Im}(H(v, w))$. Let $B = \mathrm{Im}H$. We conclude that $\mathrm{U}(H)$ is a subgroup of $\mathrm{Sp}(V, B)$.

Let $A = \mathrm{End}_{\mathbb{R}}(V)$. Here is how to pass from the general description, involving the arbitrary involution i, to the specific case of symmetric / skew-symmetric forms. By Wedderburn theory, the map L in this diagram must be an isomorphism of $\mathbb{R}$-algebras:

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{R}}(V) \arrow[r, "i"] \arrow[d, "\mathsf{T}"'] & \mathrm{End}_{\mathbb{R}}(V) \\
\mathrm{End}_{\mathbb{R}}(V^*) \arrow[ur, "L"'] &
\end{tikzcd}$$

Thus, Wedderburn etc. gives an isomorphism $h : V \to V^*$ such that $i(T) = h^{-1} T^{\mathsf{T}} h$, and thus

$$i(i(T)) = h^{-1} i(T)^{\mathsf{T}} h = h^{-1} h^{\mathsf{T}} T (\cdots)^{-1}.$$

There is a $\lambda \in \mathbb{R}$ such that $h^{\mathsf{T}} = \lambda h$ and $h = \lambda h^{\mathsf{T}}$, so that $\lambda^2 = 1$.

Now suppose that $A = \mathrm{End}_{\mathbb{C}}(V)$. First, we let $i : \mathbb{C} \to \mathbb{C}$ be the identity, and we get $G = \mathrm{O}(V, B)$ when B is symmetric (in which case $B = \mathrm{Id}_n$), and $\mathrm{Sp}(V, B)$ when B is anti-symmetric.

Second, let us take $i : \mathbb{C} \to \mathbb{C}$ be $i(z) = \overline{z}$. We get that $H : V \times V \to \mathbb{C}$ is a non-degenerate Hermitian form, and $G = \mathrm{U}(V, H)$. Choosing a basis, we have that for some $p + q = n$, the matrix of H is

$$\begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{bmatrix}$$

The compact subgroups are similar to the $\mathrm{O}(p, q)$ case.

Remark. $\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$ is a Hermitian symmetric space.

Lastly, we come to the case $K = \mathbb{H}$. Let $B : V \times V \to \mathbb{H}$ be a non-degenerate sesquilinear form. We have two cases, where $\overline{B(v,w)} = B(w,v)$ or $\overline{B(v,w)} = -B(w,v)$. In the first case, the matrix is again of the form

$$\begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{bmatrix}$$

and in the second case, the matrix is

$$\begin{bmatrix} i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & i \end{bmatrix}$$