The Riemann Zeta Function and the Distribution of Prime Numbers

Zev Chonoles

2014 – 06 – 12

Introduction

Euler was the first to study the zeta function, discovering the Euler product (Theorem 2), computing the value of $\zeta(n)$ for positive even integers and negative integers n, and from those calculations conjecturing the functional equation (Theorem 3) more than 100 years before Riemann [Ayo74]. However, Riemann's 1859 paper proving the functional equation and connecting the zeros of $\zeta(s)$ with the distribution of the prime numbers has led to its current name.

Our goal will be to introduce the main results concerning the Riemann zeta function and demonstrate its usefulness in studying the prime numbers, with the intended audience being fellow students in UChicago's quarter-long graduate complex analysis course. We will mention some of the analogies between power series and Dirichlet series. We will also go most of the way towards proving the prime number theorem, but leave the final steps unproven due to space considerations.

Let us mention some standard conventions. Any series $\sum_p$ or product $\prod_p$ is indexed over the prime numbers in increasing order. The letter Ω denotes a domain, i.e., a connected open subset $\Omega \subseteq \mathbb{C}$. Unless otherwise specified, $\log(z)$ refers to the principal branch of the logarithm, defined on the slit plane $\mathbb{C} \setminus (-\infty, 0]$. Lastly, for conciseness we sometimes speak of a subset of $\mathbb{C}$ by its defining equation or inequality – for example, the unit circle $|z| = 1$, or the half-plane $\mathrm{Re}(z) > 1$.

Basic Results

Definition. The Riemann zeta function $\zeta(z)$ is defined for $\mathrm{Re}(z) > 1$ by the formula

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Recall that the notation n^z just means $e^{z \log(n)}$. Note that the series converges absolutely in the given region: for $z = x + iy$ with $x > 1$, we have

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{n^x n^{iy}}\right| = \sum_{n=1}^{\infty} \left|\frac{e^{-iy \log(n)}}{n^x}\right| = \sum_{n=1}^{\infty} \frac{1}{n^x} \leq \int_1^{\infty} \frac{1}{t^x}\, dt = \left[\frac{t^{1-x}}{1-x}\right]_1^{\infty} = \frac{1}{x-1}.$$

Thus $\zeta(z)$ is indeed defined for $\mathrm{Re}(z) > 1$.

Definition. The von Mangoldt function $\Lambda(n)$ is a function on the positive integers defined by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This function will show up later, but for now, we are only interested in the fact that

$$\sum_{d|n} \Lambda(d) = \sum_{p} \begin{pmatrix} \text{\# of powers of } p \\ \text{that are } \leq n \end{pmatrix} \log(p) = \log(n).$$

Theorem 1. *The Riemann zeta function is analytic on* $\mathrm{Re}(z) > 1$*, and on this region, we have*

$$\zeta'(z) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^z}, \qquad \frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}.$$

Proof. Recall the following basic result of complex analysis (see, e.g., [Sch14, p.60], [Lan99, pp.156-157]):

> *Let $f_m : \Omega \to \mathbb{C}$ be a sequence of analytic functions. If f_m converges uniformly on compact subsets to a function f, then f is analytic, and for all $k \geq 1$ we have that $f_m^{(k)}$ converges to $f^{(k)}$ uniformly on compact subsets.*

In our situation, we let Ω be the half-plane $\mathrm{Re}(z) > 1$, let f be ζ, and let f_m be the partial sums

$$f_m(z) = \sum_{n=1}^{m} \frac{1}{n^z}.$$

By the Weierstrass M-test, for f_m to converge uniformly to f on a region $S \subseteq \mathbb{C}$, it will be sufficient to have a uniform bound $\sup_{z \in S} |\frac{1}{n^z}| \leq M_n$ for each $n \geq 1$ such that $\sum_{n=1}^{\infty} M_n$ converges.

Any compact subset $K \subset \Omega$ is contained within a closed half-plane $\mathrm{Re}(z) \geq c$ for some $c > 1$, so that for any $z = x + iy \in K$, we have $|\frac{1}{n^z}| = \frac{1}{n^x} \leq \frac{1}{n^c}$. Thus we have a uniform bound $\sup_{z \in K} |\frac{1}{n^z}| \leq \frac{1}{n^c}$, and we know that $\zeta(c) = \sum_{n=1}^{\infty} \frac{1}{n^c}$ converges because $c > 1$, so we conclude that f_m converges to f uniformly on K. Applying the cited result, we have that ζ is analytic on $\mathrm{Re}(z) > 1$, and that the derivatives

$$f_m'(z) = \frac{d}{dz} \sum_{n=1}^{m} \frac{1}{n^z} = \sum_{n=1}^{m} \frac{d}{dz}(e^{-z \log(n)}) = -\sum_{n=1}^{m} \frac{\log(n)}{n^z}$$

converge uniformly on compact sets to $\zeta'(z)$, so that for $\mathrm{Re}(z) > 1$,

$$\zeta'(z) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^z}.$$

This series is absolutely convergent for $\mathrm{Re}(z) > 1$, since

$$\sum_{n=1}^{\infty} \left| \frac{\log(n)}{n^z} \right| = \sum_{n=1}^{\infty} \frac{\log(n)}{n^{\mathrm{Re}(z)}} = -\zeta'(\mathrm{Re}(z))$$

which is of course finite since ζ is analytic. Finally, note that $0 \leq \Lambda(n) \leq \log(n)$ for all n, so the series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}$$

also converges absolutely for $\operatorname{Re}(z) > 1$. Thus, we can arbitrarily rearrange the doubly-indexed sum

$$\left(\sum_{k=1}^{\infty} \frac{1}{k^z}\right) \cdot \left(\sum_{d=1}^{\infty} \frac{\Lambda(d)}{d^z}\right) = \sum_{k=1}^{\infty} \sum_{d=1}^{\infty} \frac{\Lambda(d)}{(kd)^z},$$

such as by grouping into terms where $kd = n$:

$$\left(\sum_{k=1}^{\infty} \frac{1}{k^z}\right) \cdot \left(\sum_{d=1}^{\infty} \frac{\Lambda(d)}{d^z}\right) = \sum_{n=1}^{\infty}\left(\sum_{kd=n} \Lambda(d)\right) n^{-z} = \sum_{n=1}^{\infty}\left(\sum_{d|n} \Lambda(d)\right) n^{-z} = \sum_{n=1}^{\infty} \frac{\log(n)}{n^z} = -\zeta'(z)$$

thereby proving that on $\operatorname{Re}(z) > 1$, we have

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}. \qquad \square$$

We now include a brief discussion of infinite products.

Definition. Let a_n be a sequence of non-zero complex numbers. We say that the infinite product $\prod_{n=1}^{\infty} a_n$ converges (absolutely) when $\sum_{n=1}^{\infty} \log(a_n)$ converges (absolutely), where $\log(a_n)$ is allowed to mean a non-principal branch for finitely many a_n. Observe that to say $\sum_{n=1}^{\infty} \log(a_n)$ converges is just to say that the partial sums $\sum_{n=1}^{N} \log(a_n)$ have a limit, and e^z is continuous, so we define the value

$$\prod_{n=1}^{\infty} a_n = \exp\left(\sum_{n=1}^{\infty} \log(a_n)\right) = \lim_{N\to\infty} \exp\left(\sum_{n=1}^{N} \log(a_n)\right) = \lim_{N\to\infty} \prod_{n=1}^{N} a_n.$$

Observe that, as a consequence of this definition, a convergent infinite product cannot have the value 0.

Proposition. *Let a_n be a sequence of complex numbers, with $a_n \neq -1$ for all n. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges absolutely} \implies \prod_{n=1}^{\infty} (1 + a_n) \text{ converges absolutely.}$$

Proof. Recall that, unless otherwise specified, log denotes the principal branch, defined on the slit plane $\mathbb{C} \setminus (-\infty, 0]$. It is analytic on that region.

So, the function $\log(1 + z)$ is certainly holomorphic on the disk $|z| < 1$, with well-known Taylor series

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots.$$

If in fact $|z| < \frac{1}{2}$, then we have

$$|\log(1 + z)| \leq |z| + |z|^2 + |z|^3 + \cdots = \frac{|z|}{1 - |z|} \leq 2|z|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then for some N we have $|a_n| < \frac{1}{2}$ when $n \geq N$, so that $|\log(1 + a_n)| \leq 2|a_n|$ for all $n \geq N$, so that $\sum_{n=1}^{\infty} |\log(1 + a_n)|$ converges, so that $\prod_{n=1}^{\infty} (1 + a_n)$ converges absolutely. $\square$

Theorem 2 (Euler product)**.** *For* $\operatorname{Re}(z) > 1$*, we can express* $\zeta(z)$ *as the following infinite product:*

$$\zeta(z) = \prod_{p} \left(1 - \frac{1}{p^z}\right)^{-1}.$$

Proof. It is clear from the definition that a product $P = \prod_{n=1}^{\infty} a_n$ converges if and only if the product $Q = \prod_{n=1}^{\infty} a_n^{-1}$ converges, and if they do converge, we have $Q = P^{-1}$. By our proposition above, the product

$$\prod_p \left(1 - \frac{1}{p^z}\right)$$

converges absolutely for $\mathrm{Re}(z) > 1$ because

$$\sum_p \left|\frac{1}{p^z}\right| = \sum_p \frac{1}{p^{\mathrm{Re}(z)}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}} = \zeta(\mathrm{Re}(z)).$$

Therefore the product

$$\prod_p \left(1 - \frac{1}{p^z}\right)^{-1}$$

is also absolutely convergent. Since $\mathrm{Re}(z) > 1$, for any prime p we certainly can write

$$\left(1 - \frac{1}{p^z}\right)^{-1} = \sum_{k=0}^{\infty} \frac{1}{p^{kz}},$$

which is an absolutely convergent series. For any N, we can form the partial product

$$\prod_{p \leq N} \left(1 - \frac{1}{p^z}\right)^{-1} = \prod_{p \leq N} \left(\sum_{k=0}^{\infty} \frac{1}{p^{kz}}\right) = \sum_{n \in F_N} \frac{1}{n^z}$$

where F_N is the set of positive integers all of whose prime factors are $\leq N$. The rearrangement of the terms above is justified because there are finitely many series being multiplied, each of them absolutely convergent. Clearly, we have at least that $1, 2, \ldots, N \in F_N$, so that

$$\left|\zeta(z) - \prod_{p \leq N} \left(1 - \frac{1}{p^z}\right)^{-1}\right| = \left|\zeta(z) - \sum_{n \in F_N} \frac{1}{n^z}\right| \leq \sum_{m=N+1}^{\infty} \frac{1}{m^{\mathrm{Re}(z)}}.$$

But this quantity must go to 0 as $N \to \infty$ because the series $\zeta(\mathrm{Re}(z)) = \sum_{k=1}^{\infty} \frac{1}{k^{\mathrm{Re}(z)}}$ is convergent. Therefore, for $\mathrm{Re}(z) > 1$,

$$\zeta(z) = \lim_{N \to \infty} \prod_{p \leq N} \left(1 - \frac{1}{p^z}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^z}\right)^{-1}. \qquad \square$$

Corollary. *The Riemann zeta function $\zeta(z)$ has no zeros in the region* $\mathrm{Re}(z) > 1$.

Proof. A convergent infinite product cannot be zero. $\square$

Corollary. *The sum of the reciprocals of the primes, $\sum_p \frac{1}{p}$, diverges.*

Proof. As we already argued in the proof of the theorem, for any N, we have that

$$\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

Taking logarithms of both sides, we have

$$\log\left(\sum_{n=1}^{N} \frac{1}{n}\right) \leq -\sum_{p \leq N} \log\left(1 - \frac{1}{p}\right).$$

By the Taylor expansion for $\log(1+z)$, valid for $|z|<1$, we have

$$-\log\left(1-\frac{1}{p}\right)=\sum_{k=1}^{\infty}\frac{1}{kp^k},$$

and observe that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{kp^k} &\leq \frac{1}{p}+\frac{1}{2p^2}\left(\sum_{m=0}^{\infty}\frac{1}{p^m}\right)\\ &=\frac{1}{p}+\frac{1}{2p^2}\left(1-\frac{1}{p}\right)^{-1}\\ &=\frac{1}{p}+\frac{1}{2p(p-1)}\\ &<\frac{1}{p}+\frac{1}{p^2}.\end{aligned}$$

Therefore

$$\log\left(\sum_{n=1}^{N}\frac{1}{n}\right)\leq\sum_{p\leq N}\frac{1}{p}+\sum_{p\leq N}\frac{1}{p^2}\leq\sum_{p\leq N}\frac{1}{p}+\zeta(2)$$

and because $\log(\sum_{n=1}^{N}\frac{1}{n})\to\infty$ as $N\to\infty$, we must also have that $\sum_{p\leq N}\frac{1}{p}\to\infty$. □

Even with these basic results, it is easy to see that the Riemann zeta function is a powerful tool for understanding the prime numbers.

The Gamma Function

We attempt to provide a quick but rigorous introduction to the gamma function. First, we need a technical result which is termed "the differentiation lemma" in [Lan99, p.409], which we will provide without proof.

Lemma. *Let $I\subseteq\mathbb{R}$ be an interval, possibly infinite. Let $U\subseteq\mathbb{C}$ be an open set. Let $f=f(t,z)$ be a continuous function on $I\times U$. Assume that*

(i) For each compact subset $K\subset U$, the integral $\int_I f(t,z)\,dt$ is uniformly convergent for $z\in K$.

(ii) For each $t\in I$, the function $z\mapsto f(z,t)$ is analytic.

Let $F(z)=\int_I f(t,z)\,dt$. Then we have that F is analytic on U, $\frac{d}{dz}f(t,z)$ satisfies the same hypotheses as f, and

$$F'(z)=\int_I\frac{d}{dz}f(t,z)\,dt.$$

Definition. The gamma function $\Gamma(z)$ is defined for $\mathrm{Re}(z)>0$ by

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,dt.$$

Recall that t^{z-1} just means $e^{(z-1)\log(t)}$. Letting $x=\mathrm{Re}(z)$, note that we have

$$|\Gamma(z)|\leq\int_0^{\infty}|t^{z-1}e^{-t}|\,dt=\int_0^{\infty}t^{x-1}e^{-t}\,dt.$$

Certainly, there is some M such that $t^{x-1}e^{-t} \leq e^{-t/2}$ for $t \geq M$, and $\int_M^\infty e^{-t/2}\,dt$ converges. Also, we have

$$\lim_{\epsilon\to 0^+}\int_\epsilon^M t^{x-1}e^{-t}\,dt \leq \lim_{\epsilon\to 0^+}\int_\epsilon^M t^{x-1}\,dt = \lim_{\epsilon\to 0^+}\left[\frac{t^x}{x}\right]_\epsilon^M = \frac{M^x}{x}.$$

Thus $\Gamma(z)$ is indeed defined for all $\mathrm{Re}(z) > 0$. Moreover, $\Gamma(z)$ is analytic because it meets the hypotheses of the differentiation lemma, with $I = [0, \infty)$, $U =$ the half-plane $\mathrm{Re}(z) > 0$, and $f(t, z) = t^{z-1}e^z$.

Taking the definition of $\Gamma(z)$ and integrating by parts, we have for $\mathrm{Re}(z) > 0$ that

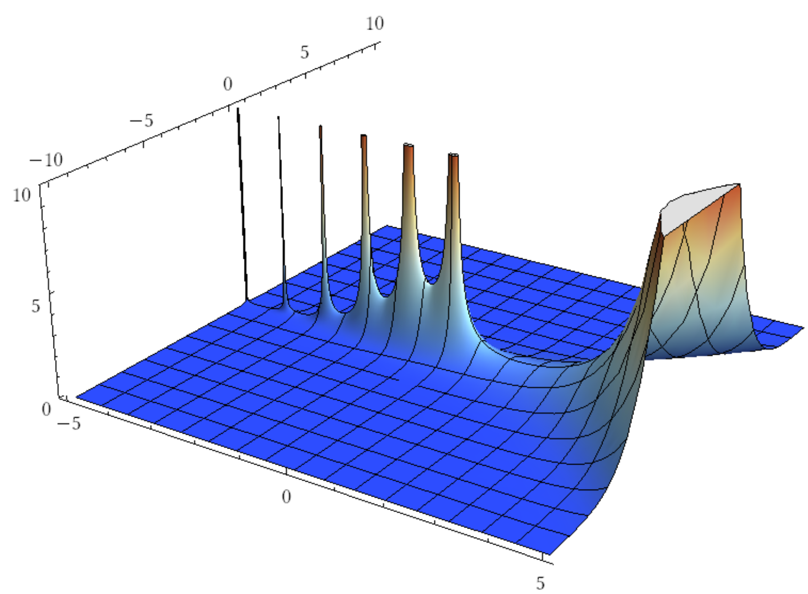
$$\Gamma(z+1) = \lim_{N\to\infty}\int_0^N t^z e^{-t}\,dt = \lim_{N\to\infty}\left[t^z(-e^{-t})\right]_0^N - \int_0^N zt^{z-1}(-e^{-t})\,dt = z\int_0^\infty t^{z-1}e^{-t}\,dt = z\Gamma(z).$$

This lets us extend $\Gamma(z)$ to a meromorphic function on the entire plane by defining, for $\mathrm{Re}(z) > -n$,

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}.$$

Since $\Gamma(z)$ is analytic on $\mathrm{Re}(z) > 0$, it is easy to see from this that the only singularities of Γ are simple poles at $z = 0, -1, -2, \ldots$, with the residue of Γ at $-n$ equal to $(-1)^n/n!$

Graphic. I created a plot in Mathematica of $|\Gamma(z)|$ on the region $|\mathrm{Re}(z)| \leq 5$, $|\mathrm{Im}(z)| \leq 10$:




```
Plot3D[Abs[Gamma[x + I y]], {x, -5, 5}, {y, -10, 10}, PlotPoints -> 25, MaxRecursion -> 5, Boxed
-> False, PlotRange -> {0, 10}, ColorFunction -> ColorData["LightTemperatureMap"], BaseStyle ->
{FontFamily -> "LinuxLibertine", FontSize -> 20}]
```

Analytic Continuation and Functional Equation of $\zeta(z)$

In this section, we will closely follow the development in [Sny02a] and [Sny02b], which is based on Riemann's own proof of these results (though not the one in his original 1859 paper; see [Con11]).

Lemma. *Define a function $\theta(z)$ on the upper half-plane $\mathbb{H}$ by the formula $\theta(z) = \sum_{m\in\mathbb{Z}} e^{\pi i m^2 z}$. Then*

$$\theta\left(-\frac{1}{z}\right) = \sqrt{-iz}\,\theta(z).$$

Proof. First, observe that the series for $\theta(z)$ converges for any $z \in \mathbb{H}$, because

$$|\theta(z)| \leq \sum_{m \in \mathbb{Z}} |e^{\pi i m^2 z}| = 1 + 2 \sum_{m=1}^{\infty} e^{-\pi m^2 \operatorname{Im}(z)} < 1 + 2 \int_0^{\infty} e^{-\operatorname{Im}(z)\, t}\, dt = 1 + \frac{2}{\operatorname{Im}(z)}.$$

Now recall the Poisson summation formula from Fourier analysis (see, e.g., [Lan93, p.244]):

> *Let $f \in C^2(\mathbb{R})$, and suppose f and its derivatives are quickly decaying – for example, it is sufficient to have $f, f', f'' \in O((1+|x|)^{-2})$. Let $\hat{f}$ denote the Fourier transform of f. Then*
>
> $$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

For any $z \in \mathbb{H}$, the function $f_z(x) = e^{\pi i x^2 z}$ meets the hypotheses of this theorem; for example,

$$|f_z(x)| = |e^{\pi i x^2 z}| = e^{-\pi x^2 \operatorname{Im}(z)}$$

decays very quickly, since $\operatorname{Im}(z) > 0$, and similarly for its derivatives. The Fourier transform of f_z is

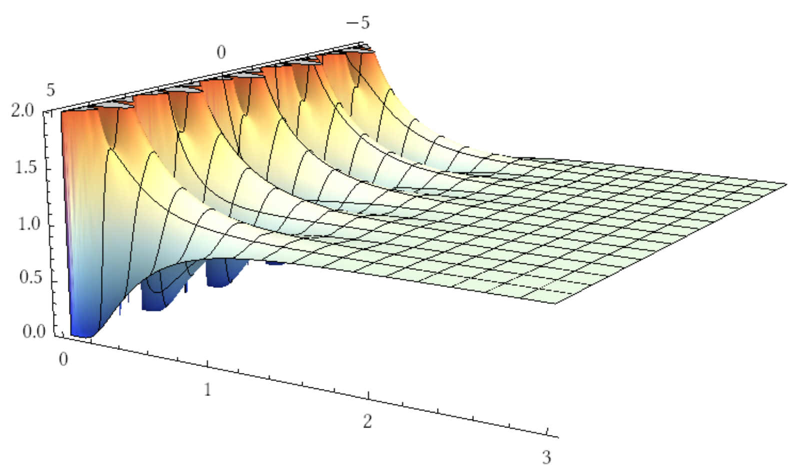
$$\hat{f}_z(x) = \frac{1}{\sqrt{-iz}} f_{1/z}(x),$$

and applying this observation to θ,

$$\theta(z) = \sum_{m \in \mathbb{Z}} f_z(m) = \sum_{m \in \mathbb{Z}} \hat{f}_z(m) = \frac{1}{\sqrt{-iz}} \sum_{m \in \mathbb{Z}} f_{1/z}(m) = \frac{1}{\sqrt{-iz}} \theta\left(-\frac{1}{z}\right).$$ □

Remark. The function θ is known as the Jacobi theta function, and this lemma essentially expresses the fact that it is a modular form of weight 1/2.

Graphic. I created a plot in Mathematica of $|\theta(z)|$ on the region $|\operatorname{Re}(z)| \leq 5$, $0 \leq \operatorname{Im}(z) \leq 3$:




```
Plot3D[Abs[EllipticTheta[3, 0, Exp[I Pi (x + I y)]]], {x, -5, 5}, {y, 0, 3}, PlotPoints
-> 25, MaxRecursion -> 5, Boxed -> False, PlotRange -> {0, 2}, ColorFunction ->
ColorData["LightTemperatureMap"], BaseStyle -> {FontFamily -> "LinuxLibertine", FontSize -> 20}]
```

Theorem 3. *There exists a meromorphic continuation of $\zeta(z)$ to the entire plane, analytic everywhere except for a simple pole at $z = 1$, and it satisfies the functional equation*

$$\Gamma(\tfrac{z}{2})\pi^{-z/2}\zeta(z) = \Gamma(\tfrac{1-z}{2})\pi^{-(1-z)/2}\zeta(1-z).$$

More simply, letting $\xi(z) = \pi^{-z/2}\Gamma(\frac{z}{2})\zeta(z)$, we have that $\xi(z) = \xi(1-z)$.

Proof. Recall that the definition of the gamma function for $\mathrm{Re}(z) > 0$ is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$$

Making a change of variables $t \mapsto n^2 \pi t$, we obtain

$$n^{-2z} \pi^{-z} \Gamma(z) = \int_0^\infty t^{z-1} e^{-\pi n^2 t} \, dt.$$

We would like to sum both sides over $n \geq 1$, and interchange the sum and integral. However, we need to check that this is allowed. Interchanging a sum and integral is just a special case of Fubini's theorem. Letting $x = \mathrm{Re}(z)$, Fubini's theorem requires us to check that

$$\sum_{n=1}^\infty \int_0^\infty |t^{z-1} e^{-\pi n^2 t}| \, dt = \sum_{n=1}^\infty \int_0^\infty t^{x-1} e^{-\pi n^2 t} \, dt = \sum_{n=1}^\infty n^{-2x} \pi^{-x} \Gamma(x) = \zeta(2x) \pi^{-x} \Gamma(x) < \infty$$

which is true for $x > \frac{1}{2}$. Thus, in the region $\mathrm{Re}(z) > \frac{1}{2}$, we have

$$\zeta(2z) \pi^{-z} \Gamma(z) = \int_0^1 \frac{1}{2} (\theta(it) - 1) t^z \, \frac{dt}{t}.$$

Observe that we have rearranged the powers of t slightly; however it is of no real consequence, other than to make our computations slightly neater below. Also, this is the traditional form of a Mellin transform.

Now we have that

$$\begin{aligned}
\zeta(2z)\pi^{-z}\Gamma(z) &= \int_0^\infty \frac{1}{2}(\theta(it) - 1) t^z \, \frac{dt}{t} \\
&= \int_0^1 \frac{1}{2}(\theta(it) - 1) t^z \, \frac{dt}{t} + \int_1^\infty \frac{1}{2}(\theta(it) - 1) t^z \, \frac{dt}{t} \\
\text{(change of variables)} \quad &= \int_1^\infty \frac{1}{2}(\theta(-1/it) - 1) t^{-z} \, \frac{dt}{t} + \int_1^\infty \frac{1}{2}(\theta(it) - 1) t^z \, \frac{dt}{t} \\
\text{(lemma about } \theta\text{)} \quad &= \int_1^\infty \frac{1}{2}(t^{1/2}\theta(it) - 1) t^{-z} \, \frac{dt}{t} + \int_1^\infty \frac{1}{2}(\theta(it) - 1) t^z \, \frac{dt}{t} \\
&= \int_1^\infty \frac{1}{2}(\theta(it) - 1)(t^{1/2-z} + t^z) \, \frac{dt}{t} + \int_1^\infty \frac{1}{2}(-t^{1/2-z} - t^{-z}) \, \frac{dt}{t} \\
&= -\frac{1}{2(\frac{1}{2} - z)} - \frac{1}{2z} + \int_1^\infty \frac{1}{2}(\theta(it) - 1)(t^{1/2-z} + t^z) \, \frac{dt}{t}
\end{aligned}$$

But the right side is defined, and indeed analytic, on all of $\mathbb{C}$ except for its simple poles at $z = 0$ and $z = \frac{1}{2}$. This is by the "differentiation lemma" cited earlier. The simple pole at $z = 0$ is contributed from the factor of $\Gamma(z)$ on the left side, so the above equation defines an analytic continuation of $\zeta(z)$ to the entire plane except for a simple pole at $z = 1$, which by construction satisfies this functional equation since the left side is symmetric under changing $z \mapsto \frac{1}{2} - z$. $\square$

Observe that the functional equation immediately implies that $\zeta(z)$ has simple zeros at the negative even integers $-2, -4, \ldots$ to cancel out the simple poles of $\Gamma(z)$, and moreover, that there are no other zeros outside of the region $0 \leq \mathrm{Re}(z) \leq 1$ because we proved that $\zeta(z)$ has no zeros for $\mathrm{Re}(z) > 1$.

Remark. As we saw, it is somewhat more natural to express the functional equation as a statement about

$$\xi(z) = \pi^{-z/2}\Gamma(\tfrac{z}{2})\zeta(z)$$

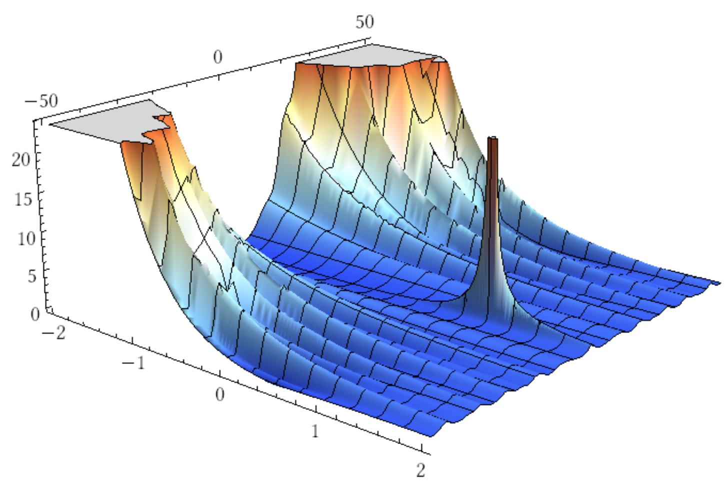
than about $\zeta(z)$ itself. This function is often referred to as the "completed" Riemann zeta function, and in a certain precise sense, the extra factor

$$\pi^{-z/2}\Gamma(\tfrac{z}{2}) = \int_{-\infty}^{\infty} |x|^z e^{-\pi x^2} \frac{dx}{|x|}$$

is what should be included in the Euler product (Theorem 2) to account for the Archimedean place of $\mathbb{Q}$; the non-Archimedean places of $\mathbb{Q}$ are precisely the p-adic absolute values, for each prime number p, and their contributions in the Euler product are precisely the factors $(1 - \frac{1}{p^z})^{-1}$.

See [Arn09] and [Shu12] for more information.

Graphic. I created a plot in Mathematica of $|\zeta(z)|$ on the region $|\mathrm{Re}(z)| \le 2$, $|\mathrm{Im}(z)| \le 50$:




```
Plot3D[Abs[Zeta[x + I y]], {x, -2, 2}, {y, -50, 50}, PlotPoints -> 15, MaxRecursion -> 5, Boxed ->
False, ColorFunction -> ColorData["LightTemperatureMap"], BaseStyle -> {FontFamily -> "LinuxLibertine",
FontSize -> 20}]
```

The simple pole of $\zeta(z)$ at $z = 1$ is quite apparant in this image.

Comparing Dirichlet Series and Power Series

Before we move on to the prime number theorem, we will need a technical result known as Perron's formula. Due to lack of space we will not include a proof of this result, but to compensate, we will motivative it by discussing an analogy between power series and Dirichlet series which is quite interesting in its own right.

In this section, the results are all taken from [Apo76, Ch.11].

Definition. Given a function $f : \mathbb{N} \to \mathbb{C}$, a series of the form $\sum_{n=1}^{\infty} \frac{f(n)}{n^z}$ is known as a Dirichlet series.

Theorem. *There is some $\sigma_c \in [-\infty, \infty]$, called the abscissa of convergence, such that $\sum_{n=1}^{\infty} f(n)/n^z$ converges for all $\mathrm{Re}(z) > \sigma_c$ and does not converge for all $\mathrm{Re}(z) < \sigma_c$.*

Theorem. *There is some $\sigma_a \in [-\infty, \infty]$, called the abscissa of absolute convergence, such that $\sum_{n=1}^{\infty} f(n)/n^z$ converges absolutely for all $\mathrm{Re}(z) > \sigma_a$ and does not converge absolutely for all $\mathrm{Re}(z) < \sigma_a$.*

Theorem. *For any Dirichlet series with σ_c finite, we have $0 \le \sigma_a - \sigma_c \le 1$.*

Remark. An example of the worst-case scenario is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$, for which $\sigma_c = 0$ and $\sigma_a = 1$.

Theorem. *A Dirichlet series converges uniformly on compact subsets of the half-plane $\mathrm{Re}(z) > \sigma_c$.*

So we see that

	power series	Dirichlet series
region of convergence	disk	right half-plane
absolute convergence vs. convergence	$\Longleftrightarrow$	$\Longrightarrow$
analytic function when convergent	yes	yes

Now we come to Perron's formula, and an attendant lemma.

Lemma. *If $c > 0$, define $\int_{c-i\infty}^{c+i\infty}$ to mean $\lim_{T\to\infty} \int_{c-iT}^{c+iT}$. For any real number $a > 0$, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} a^z \frac{dz}{z} = \begin{cases} 1 & \text{if } a > 1, \\ \frac{1}{2} & \text{if } a = 1, \\ 0 & \text{if } 0 < a < 1. \end{cases}$$

Theorem 4 (Perron's formula)**.** *Let $F(z) = \sum_{n=1}^{\infty} f(n)/n^z$ be absolutely convergent for $\mathrm{Re}(z) > \sigma_a$; let $c > 0$ and $x > 0$ arbitrary. Then if $Re(z) > \sigma_a - c$ we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z+t)\frac{x^t}{t}\, dt = \sum_{n\le x}{}^{*} \frac{f(n)}{n^z}$$

where $\sum^*$ means that the last term in the sum must be multiplied by $\frac{1}{2}$ when x is an integer.

These results fit into the analogy, as explained in [Siu05]:

> *The Perron formula for Dirichlet series plays the same role as the Cauchy formula for power series.*
>
> *... The domain of convergence of a power series is a disk, whereas the domain of convergence of a Dirichlet series is a left half-plane. So the integration in the Cauchy formula for a power series is a circle, whereas the integration in the Perron formula is a vertical line. There is a difference – the Perron formula gives a partial sum of the Dirichlet series instead of the whole series. It corresponds to the following truncated form of the Cauchy formula:*

$$\begin{aligned} \sum_{n=1}^{m} a_n (z-c)^n &= \frac{1}{2\pi i}\int_{|w-c|=r} f(w)\left(\sum_{n=1}^{m}\frac{(z-c)^n}{(w-c)^{n+1}}\right) dw \\ &= \frac{1}{2\pi i}\int_{|w-c|=r} f(w)\cdot \frac{1-\left(\frac{z-c}{w-c}\right)^{m+1}}{w-z}\, dw \end{aligned}$$

… The lemma corresponds to

$$\frac{1}{2\pi i}\int_{|z|=r} z^n\,dz = \begin{cases} 1 & \text{if } n=-1,\\ 0 & \text{if } n\neq -1\end{cases}$$

for power series, and is used to pick out one coefficient in a power series.

In [Eri08], Perron's formula is motivated in a way more directly relevant to the prime number theorem:

Given a reasonably well-behaved (e.g., not exponentially growing) function $f:\mathbb{N}\to\mathbb{C}$, the value of

$$F(x)=\sum_{n\le x} f(n)$$

is intimately tied to the poles of

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

via Perron's formula.

The Prime Number Theorem

Definition. The prime-counting function $\pi(x)$ is defined for $x>0$ to be $\pi(x)=\#\{\text{prime numbers} \le x\}$.

The famous prime number theorem, which we will discuss in this section, asserts that

$$\pi(x)\sim\frac{x}{\log(x)}.$$

Definition. The second Chebyshev function $\psi(x)$ is defined by

$$\psi(x)=\sum_{n\le x}\Lambda(n).$$

This function of course has jump discontinuities at prime powers, and for many purposes it is useful to consider a slightly "smoothed" version $\psi_0(x)$, defined by

$$\psi_0(x)=\frac{1}{2}\left(\sum_{n<x}\Lambda(n)+\sum_{n\le x}\Lambda(n)\right)=\begin{cases}\psi(x) & \text{if } \psi \text{ is continuous at } x,\\ \frac{\psi(x^+)+\psi(x^-)}{2} & \text{if } \psi \text{ is not continuous at } x.\end{cases}$$

In [Ing90, p.13], we find an interesting remark:

It happens… that, of the three functions π, ϑ, and ψ, the one which arises most naturally from the analytical point of view is the one most remote from the original problem, namely ψ… This is a complication which seems inherent in the subject, and the reader should familiarize himself at the outset with the function ψ, which is to be regarded as the fundamental one.

Proposition. *$\psi(x)\sim x$ if and only if $\pi(x)\sim\frac{x}{\log(x)}$.*

Theorem 5 (von Mangoldt's explicit formula for ψ_0)**.**

$$\psi_0(x)=x-\sum_{\substack{\zeta(\rho)=0\\ 0<\mathrm{Re}(\rho)<1}}\frac{x^\rho}{\rho}-\log(2\pi)-\frac{1}{2}\log(1-x^{-2}).$$

Proof. Let $f(n) = \Lambda(n)$, and let $z = 0$. We have that $F(z) = \sum_{n=1}^{\infty} f(n)/n^z = -\frac{\zeta(z)}{\zeta'(z)}$ by Theorem 1. Applying Perron's formula (Theorem 4), we get that for $\mathrm{Re}(z) > 1$ and some $c > 1$,

$$\psi_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(z)}{\zeta(z)}\right) \frac{x^z}{z}\, dz.$$

However, the integrand is a meromorphic function on the plane by our continuation of $\zeta(z)$ in Theorem 3, so we can move the contour left, outside the region $\mathrm{Re}(z) > 1$, as long as we account for any residues we pick up at poles of the integrand along the way.

What are the poles we encounter as we move the contour further and further left, i.e., $c \to -\infty$?

(We mentioned immediately following Theorem 3 its help in understanding the zeros of $\zeta(z)$, and we use that work here.)

- a pole at $z = 1$ caused by the factor $-\frac{\zeta'(z)}{\zeta(z)}$ of the integrand, with residue x
- a pole at each non-trivial zero $z = \rho$ of ζ in the strip $0 \le \mathrm{Re}(z) \le 1$ caused by the factor $-\frac{\zeta'(z)}{\zeta(z)}$ of the integrand, with residue $-\frac{mx^\rho}{\rho}$ where m is the order to which ζ vanishes at ρ
- a pole at $z = 0$ caused by the factor $\frac{x^z}{z}$ of the integrand, with residue $-\frac{\zeta'(0)}{\zeta(0)}$
- a pole at each trivial zero $z = -2, -4, \ldots$ of ζ, caused by the factor $-\frac{\zeta'(z)}{\zeta(z)}$ of the integrand, with residue $\frac{1}{2mx^{2m}}$ contributed by $z = -2m$.

One can prove that the integrand goes to 0 as $c \to -\infty$, so in fact we have that

$$\psi_0(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ 0 \le \mathrm{Re}(\rho) \le 1 \\ \text{with mult.}}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \underbrace{\sum_{m=1}^{\infty} \frac{1}{2mx^{2m}}}_{\frac{1}{2}\log(1-x^{-2})}$$

Lastly, one can prove that $-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi)$. □

By proving that $\zeta(z)$ has no zeros on the line $\mathrm{Re}(z) = 1$ (which by the functional equation also implies no zeros on the line $\mathrm{Re}(z) = 0$), the sum over the non-trivial zeros ρ decays fast enough as $x \to \infty$ to prove that $\psi_0(x) \sim x$, which is equivalent to $\pi(x) \sim \frac{x}{\log(x)}$, the prime number theorem.

References

[Apo76] Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[Arn09] Peter Arndt, *Why does the Gamma-function complete the Riemann Zeta function?*, MathOverflow, (link), 2009.

[Ayo74] Raymond Ayoub, *Euler and the zeta function*, The American Mathematical Monthly **81** (1974), no. 10, pp. 1067–1086.

[Con11] Keith Conrad, *How does one motivate the analytic continuation of the Riemann zeta function?*, MathOverflow, (link), 2011.

[Dav00] Harold Davenport, *Multiplicative Number Theory*, 3rd ed., Springer-Verlag, New York, 2000.

[Eri08] Carl Erickson, *Prime Numbers and the Riemann Hypothesis,* PROMYS Minicourse, (link), 2008.

[Ing90] Albert Ingham, *The Distribution of Prime Numbers*, Cambridge University Press, Cambridge, 1990.

[Lan93] Serge Lang, *Real and Functional Analysis*, 3rd ed., Springer-Verlag, New York, 1993.

[Lan99] ———, *Complex Analysis*, 4th ed., Springer-Verlag, New York, 1999.

[Sch14] Wilhelm Schlag, *A course in complex analysis and Riemann surfaces*, unpublished draft, 2014.

[Shu12] Jerry Shurman, *Local Factors of Zeta Functions*, (link), 2012.

[Siu05] Yum-Tong Siu, *Perron Formula Without Error Estimates*, (link), 2005.

[Sny02a] Noah Snyder, *Lecture #3: A Review of Fourier Analysis*, (link), 2002.

[Sny02b] ———, *Lecture #4: The Analytic Continuation and Functional Equation of Riemann's Zeta Function*, (link), 2002.