Math 254 – Number Theory 2

Lectures by Joe Silverman
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Introduction

Math 254 is one of the courses offered for mathematics graduate students at Brown University. It is the second of two courses in the year-long number theory sequence. I took these notes while taking the course as a junior.

The notes are handwritten because this was before I started live-TeXing. I may eventually get around to typing these notes properly.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to chonoles@math.uchicago.edu.

Lecture 1 (2011-01-26)

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Topics
- Class field theory < Global, Local
- Group Cohomology (especially Galois cohomology)
- Elliptic Curves < Mordell-Weil theorem, CM

References: see web page
in particular Lang Algebraic Number Theory
Serre Local fields

No class April 27, 29, May 2, so make-up class during reading period, May 4, 6, 9

Class Field Theory

Study of abelian extensions L/K - describe them in terms of the arithmetic of K

Example: Kronecker-Weber theorem
Let $L/\mathbb{Q}$ be an abelian extension. Then
$L \subset \mathbb{Q}(\mu_N)$ for some N ($\mu_N = \{\text{roots of } x^N - 1\}$)

Review of Algebraic Number Theory (global)

$K/\mathbb{Q}$ a number field, $\mathcal{O}_K$ its ring of integers

Prop. $\mathcal{O}_K$ is a Dedekind domain

hence ideals have unique factorization

Given an ideal $\mathfrak{a} \subset \mathcal{O}_K$, $\mathfrak{a} = \prod \mathfrak{p}^{e_\mathfrak{p}}$

$e_\mathfrak{p} = \mathrm{ord}_\mathfrak{p}(\mathfrak{a}) = v_\mathfrak{p}(\mathfrak{a})$

$I_K = \{\text{fractional ideals}\} = \{\text{finitely generated } \mathcal{O}_K\text{-modules (whose } K\text{-span is } K) \text{ of } K\}$

(fractional ideals: implicitly non-zero)

$\mathfrak{a}^{-1} = \{x \in K^\times \mid x\mathfrak{a} \subset \mathcal{O}_K\}$

I_K is a group

$P_K = \{\text{principal fractional ideals}\} = \{a\mathcal{O}_K \mid a \in K^\times\}$

$C_K = I_K / P_K$ = ideal class group of K

Thm. C_K is finite. $h_K = |C_K|$ = class number of K

Thm (Dirichlet). $\mathcal{O}_K^\times$ is finitely generated.
In particular, $\mathcal{O}_K^\times \cong \mu_K \times \mathbb{Z}^{r_1+r_2-1}$

(μ_K: roots of unity in K)

$r_1 = \#$ of real embeddings of K
$r_2 = \frac{1}{2}(\#$ of complex embeddings of $K)$

Galois group acts on $\mathcal{O}_K^\times$, and in particular $\mathbb{Z}^{r_1+r_2-1}$
choosing a basis, for each $\sigma \in G_{K/\mathbb{Q}}$ we get a matrix, hence a representation

$$1 \to \mathcal{O}_K^\times \to K^\times \to \mathcal{P}_K \to 1$$

$$1 \to \mathcal{P}_K \to I_K \to C_K \to 1$$

or $1 \to \mathcal{O}_K^\times \to K^\times \to I_K \to C_K \to 1$

$\alpha \mapsto \alpha\mathcal{O}_K$

Let $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be a finite set of primes

(Silverman will occasionally abuse notation and use $\mathrm{Spec}(\mathcal{O}_K) = \{\text{non-zero prime ideals}\}$)

$\mathcal{O}_S = \mathcal{O}_{K,S}$ = ring of S-integers =
$\{\alpha \in K^\times \mid \mathrm{ord}_\mathfrak{p}(\alpha) \geq 0 \text{ for all } \mathfrak{p} \notin S\} \cup \{0\}$

Thm, $\mathcal{O}_{K,S}^\times \cong \mu_K \times \mathbb{Z}^{r_1+r_2+|S|-1}$

Example. $K = \mathbb{Q}$, $\mathbb{Z}^\times = \mu_\mathbb{Q} = \{\pm 1\}$, $\mathbb{Z}_{\{2,3\}}^\times = \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]^\times =$
$\{\pm 2^i 3^j \mid i,j \in \mathbb{Z}\}$

Extension fields

L/K, $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)$, $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_r^{e_r}$, $e_i \geq 1$
$n = [L:K]$ ← ramification index

we write $\mathfrak{P}_i \mid \mathfrak{p}$. Also, $e_\mathfrak{p} = e(\mathfrak{P}/\mathfrak{p}) = \mathrm{ord}_\mathfrak{P}(\mathfrak{p}\mathcal{O}_L)$.
$(\Leftrightarrow \mathfrak{p}\mathcal{O}_L \subset \mathfrak{P}_i)$

Let $\mathfrak{P} \mid \mathfrak{p}$. $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ induces $\mathcal{O}_K/\mathfrak{p} \hookrightarrow \mathcal{O}_L/\mathfrak{P}$.

Silverman's notation: $\mathbb{F}_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$, $\mathbb{F}_\mathfrak{P} = \mathcal{O}_L/\mathfrak{P}$

$f(\mathfrak{P}/\mathfrak{p}) = [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]$ is the residue field degree.

Thm. Given $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$, $\sum_{i=1}^{r} e_i f_i = n$

If L/K is Galois, $e(\mathfrak{P}_i/\mathfrak{p})$ all the same,
$f(\mathfrak{P}_i/\mathfrak{p})$ all the same,
hence $\mathfrak{p}\mathcal{O}_L = (\mathfrak{P}_1 \cdots \mathfrak{P}_r)^e$, $ref = n$

$\mathfrak{p}$ is unramified if all $e(\mathfrak{P}_i/\mathfrak{p}) = 1$, ramified otherwise

$\mathfrak{p}$ is totally ramified if $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}^n$, i.e. there is a prime $\mathfrak{P} \mid \mathfrak{p}$ with $e(\mathfrak{P}/\mathfrak{p}) = n$ (hence $r = 1$, $f = 1$)

$\mathfrak{p}$ is inert if $\mathfrak{p}\mathcal{O}_L$ is prime, i.e. $e(\mathfrak{P}/\mathfrak{p}) = 1$, $f(\mathfrak{P}/\mathfrak{p}) = n$

$\mathfrak{p}$ is totally split if $e(\mathfrak{P}_i/\mathfrak{p}) = 1$, $f(\mathfrak{P}_i/\mathfrak{p}) = 1$,
$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1 \cdots \mathfrak{P}_r$

Thm. $\{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K) \mid \mathfrak{p}$ ramified in $L\}$ finite

Given $x_1, \ldots, x_n \in L$, $\mathrm{disc}(x_1, \ldots, x_n) = \det(\mathrm{Tr}_{L/K}(x_i x_j))$
$= \det(\sigma_i(x_j))^2$,

$\mathrm{Tr}_{L/K} = \sum_{\substack{\sigma : L \hookrightarrow \bar{K} \\ \sigma|_K = \mathrm{id}}} \sigma(x)$

$\mathfrak{p}$ ramified $\Leftrightarrow \mathfrak{p} \mid \mathrm{Disc}$

Lecture 2 (2011-01-28)

$\alpha \in \mathcal{O}_L$, or $\alpha \in I_L$,

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$\mathrm{Disc}(\alpha) = \langle \mathrm{Disc}(x_1, \dots, x_n) \rangle_{\alpha = (x_1, \dots, x_n)} \in I_L$

Thm. $\mathfrak{p}$ ramifies $\iff \mathfrak{p} \mid \underbrace{\mathrm{Disc}_{L/K}(\mathcal{O}_L)}_{d_{L/K}}$

$d_K = \mathrm{Disc}_{K/\mathbb{Q}}(\mathcal{O}_K)$

The Norm $N_{L/K}(\mathfrak{P}) = \mathfrak{p}^{f(\mathfrak{P}|\mathfrak{p})}$

Extend to I_L, $N_{L/K}(\prod \mathfrak{P}_i^{a_i}) = \prod N_{L/K}(\mathfrak{P}_i)^{a_i}$

If L/K Galois, $N_{L/K}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\alpha)$

If $\alpha \in L$, $N_{L/K}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\alpha)$, $N_{L/K}((\alpha)) = (N_{L/K}(\alpha))$

$N_{E/K} = N_{L/K} \circ N_{E/L}$

Galois extensions

L/K, $G_{L/K} = \mathrm{Gal}(L/K)$, $\mathfrak{P} \mid \mathfrak{p}$

Decomposition group: $D(\mathfrak{P}|\mathfrak{p}) = \{\sigma \in G_{L/K} \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$

$\sigma \in D(\mathfrak{P}|\mathfrak{p})$ induces an automorphism of $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}}$

So we get a homomorphism $D(\mathfrak{P}|\mathfrak{p}) \to \underbrace{\mathrm{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})}_{\text{cyclic}}$

The kernel of this map is the inertia group

$$I(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in D(\mathfrak{P}/\mathfrak{p}) \mid \sigma(a) \equiv a \bmod \mathfrak{P} \text{ for all } a \in \mathcal{O}_L\}$$

$\mathfrak{P}/\mathfrak{p}$ unramified iff $I(\mathfrak{P}/\mathfrak{p}) = 1$, and

furthermore, $|I(\mathfrak{P}/\mathfrak{p})| = e(\mathfrak{P}/\mathfrak{p})$

Thm. $G_{L/K}$ acts transitively on the $\mathfrak{P}_i/\mathfrak{p}$.

$D(\mathfrak{P}_i/\mathfrak{p})$ and $D(\mathfrak{P}_j/\mathfrak{p})$ are all conjugate

$N\mathfrak{p} = |\mathbb{F}_\mathfrak{p}|$, $N\mathfrak{P} = |\mathbb{F}_\mathfrak{P}|$

N means norm down to $\mathbb{Q}$

$\mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p})$ is cyclic, generated by Frobenius

$a \longmapsto a^{N\mathfrak{p}}$

$\phi_\mathfrak{p}$ or $\mathrm{Frob}_\mathfrak{p}$

$|\mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p})| = f(\mathfrak{P}/\mathfrak{p})$

$D(\mathfrak{P}/\mathfrak{p})/I(\mathfrak{P}/\mathfrak{p}) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{F}_\mathfrak{P}/\mathbb{F}_\mathfrak{p})$, so that

if $\mathfrak{p}$ is unramified, $I(\mathfrak{P}/\mathfrak{p}) = 1$, hence there is a unique element of $D(\mathfrak{P}/\mathfrak{p})$ that induces $\phi_\mathfrak{p}$ on $\mathbb{F}_\mathfrak{P}$. This is denoted $\left(\frac{\mathfrak{P}/\mathfrak{p}, L/K}{}\right) \in D(\mathfrak{P}/\mathfrak{p})$

Called the Artin symbol.

Lemma. $\mathfrak{P}_1, \mathfrak{P}_2 \mid \mathfrak{p} \Rightarrow (\mathfrak{P}_1/\mathfrak{p}, L/K)$ and $(\mathfrak{P}_2/\mathfrak{p}, L/K)$ one conjugate

So $\mathfrak{p}$ determines a conjugacy class $(\mathfrak{p}, L/K) \subset G_{L/K}$

If L/K abelian, $(\mathfrak{p}, L/K) \in G_{L/K}$ "Box that and put stars around it".

Artin Map (L/K abelian)

$$(\cdot, L/K): I_K^{unr} \longrightarrow G_{L/K}$$
$$(\textstyle\prod \mathfrak{p}_i, L/K) \to \prod (\mathfrak{p}_i, L/K)$$

Thm. (part of class field theory)

L/K abelian (and K is totally imaginary), then

$\exists$ an ideal $\mathfrak{C}$ (conductor) $\subset \mathcal{O}_K$ such that

Let $\alpha \in \mathcal{O}_K$, $\alpha \equiv 1 \bmod \mathfrak{C}$. Then $(\alpha\mathcal{O}_K, L/K) = 1$.

(Still not quite true.)

Example. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

$(a\mathbb{Z}, K/\mathbb{Q}) = \prod (p_i\mathbb{Z}, K/\mathbb{Q})$

but $(p\mathbb{Z}, K/\mathbb{Q})$ needs to raise things to pth powers, so

$(p\mathbb{Z}, K/\mathbb{Q})(\zeta_n) \equiv \zeta_n^p \bmod \mathfrak{P}$

Prop. $\mu_n \to \mathcal{O}_K/\mathfrak{p} = \mathbb{Z}[\zeta_n]/\mathfrak{p}$ is injective iff $p \nmid n$.

(HW #1) ↗

Thus we actually have $(p\mathbb{Z}, K/\mathbb{Q})(\zeta_n) = \zeta_n^p$

so $(a\mathbb{Z}, K/\mathbb{Q})(\zeta_n) = \zeta_n^a$, so $(a\mathbb{Z}, K/\mathbb{Q}) = 1$
$\Leftrightarrow a \equiv 1 \bmod n$

So n is the conductor of the cyclotomic extension

The roots of unity were helping us keep track of things

Dirichlet's Theorem

$$\{\text{primes } p \equiv 1 \bmod n\} = \{\text{primes } p \mid (p\mathbb{Z}, \mathbb{Q}(\zeta_n)/\mathbb{Q}) = 1\}$$

Lecture 3 (2009-01-31)

Office Hours (email in advance) 2011-1-31 (3)
Mon 2–3 Fri 9:30–10:30

$(\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\sim} Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$
$a \longmapsto (\sigma_a : \zeta_n \to \zeta_n^a)$ and $(p, \mathbb{Q}(\zeta_n)/\mathbb{Q}) = \sigma_p$

So $\{p \mid p \equiv a \bmod n\} = \{p : (p, \mathbb{Q}(\zeta_n)/\mathbb{Q}) = \sigma_a\}$

Generalize to arbitrary number fields L/K (Galois)

$\mathfrak{p} \in Spec(\mathcal{O}_K)$, unramified in L, then $(\mathfrak{p}, L/K)$ is a conjugacy class in $G_{L/K}$

So fix a conjugacy class $C \subset G_{L/K}$, and look at
$\{\mathfrak{p} \in Spec(\mathcal{O}_K) \mid (\mathfrak{p}, L/K) = C\}$

$$\pi_K(x, C) = \#\{\mathfrak{p} \in Spec(\mathcal{O}_K) \mid (\mathfrak{p}, L/K) = C, N_{\mathbb{Q}}^K(\mathfrak{p}) \le x\}$$

$$\pi_K(x) = \#\{\mathfrak{p} \in Spec(\mathcal{O}_K) \mid N_{\mathbb{Q}}^K(\mathfrak{p}) \le x\}$$

A subset $S \subset Spec(\mathcal{O}_K)$ has natural density

$$\delta(S) = \lim_{x \to \infty} \frac{\#\{\mathfrak{p} \in S \mid N\mathfrak{p} \le x\}}{\pi_K(x)} \quad \text{(if limit exists)}$$

Chebotarov Density Theorem

$$\delta(C) = \frac{\#C}{\#G_{L/K}}$$

Given some $\mathcal{P} \subset \mathrm{Spec}(\mathbb{Z})$

$$\sum_{p} \frac{1}{p^s} \sim -\log(s-1) \quad \text{as} \quad s \to 1^+$$

so analytic density of $\mathcal{P}$ is

$$\delta^{an}(\mathcal{P}) = \lim_{s \to 1} \frac{1}{-\log(s-1)} \sum_{p \in \mathcal{P}} \frac{1}{p^s}$$

Thm. If $\delta^{nat}(\mathcal{P})$ exists, then so does $\delta^{an}(\mathcal{P})$, and $\delta^{nat}(\mathcal{P}) = \delta^{an}(\mathcal{P})$.

New density: $\delta^{an-2}(\mathcal{P}) = \lim_{s \to 1^+} (s-1) \sum_{p \in \mathcal{P}} \frac{\log(p)}{p^s}$

HW 2: $\sum_{p} \frac{\log(p)}{p^s} \sim \frac{1}{s-1}$ as $s \to 1^+$

Hint: differentiate $\log(\zeta(s))$

Special case: Fix a prime q, and consider $\{p \mid p \equiv a \bmod q\}$

$$\sum_{p \equiv a \bmod q} \frac{\log(p)}{p^s} = \sum_{p} \begin{pmatrix} 1 \text{ if } p \equiv a \bmod q \\ 0 \text{ if } p \not\equiv a \bmod q \end{pmatrix} \frac{\log(p)}{p^s}$$

How can we define this non-step-function like?, Find a sum with this property - then we can switch the sums,

Characters for abelian groups

G finite abelian group, $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$

Prop. a) $\hat{G} \cong G$ (non-canonical)
b) $\hat{\hat{G}} \cong G$ (canonical)

HW 3: G abelian, $\#G = \infty$, then $G \hookrightarrow \hat{\hat{G}}$

Pf in case G is cyclic (in particular $G = (\mathbb{Z}/q\mathbb{Z})^\times$)

$G \cong \mathbb{Z}/n\mathbb{Z}$, then define a map

$F: G \to \hat{G}$
$a \mapsto (k \mapsto \zeta_n^{ak})$

Injective: $F(a) = 1 \Rightarrow$ $\zeta_n^{ak} = 1$ for all k, hence $\zeta_n^a = 1$, hence $a \equiv 0 \bmod n$

Surjective $\chi \in \hat{G}$, then $\chi(1) = \zeta_n^b$ for some b,
so $\chi(k) = \zeta_n^{bk}$, so $\chi = F(b)$

b) $G \hookrightarrow \hat{\hat{G}} = \mathrm{Hom}(\hat{G}, \mathbb{C}^\times)$
$a \mapsto (\chi \mapsto \chi(a))$ and $\#G = \#\hat{G} = \#\hat{\hat{G}}$ by a) twice

Orthogonality

G a finite abelian group, χ_0 the trivial character

a) $\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} \#G & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$

b) $\sum_{a \in G} \chi(a) = \begin{cases} \#G & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases}$

Pf. a) $g \in G$, $g \neq 1$, then $\exists \gamma \in \hat{G}$ with $\gamma(g) \neq 1$

such a γ exists because $G \cong \hat{G}$

(if $\gamma(g) = 1$, then γ reduces to element of $\widehat{(G/\langle g \rangle)}$)

$$\gamma(g) \sum_{\chi \in \hat{G}} \chi(g) = \sum_{\chi \in \hat{G}} (\gamma\chi)(g) = \sum_{\chi \in \hat{G}} \chi(g)$$

$\gamma(g) \neq 1$ hence $\sum_{\chi \in \hat{G}} \chi(g) = 0$

b) similar

So let $G_q = (\mathbb{Z}/q\mathbb{Z})^\times$ cyclic of order $q-1$

$$\sum_{\chi \in \hat{G}_q} \chi(b) = \begin{cases} q-1 & \text{if } b \equiv 1 \bmod q \\ 0 & \text{if } b \not\equiv 1 \bmod q \end{cases}$$

$$\frac{1}{q-1} \sum_{\chi \in \hat{G}_q} \chi(a^{-1}p) = \begin{cases} 1 & \text{if } p \equiv a \bmod q \\ 0 & \text{if } p \not\equiv a \bmod q \end{cases}$$

$\chi(a)^{-1}\chi(p) = \overline{\chi(a)}\,\chi(p)$

$$\sum_{p \equiv a \bmod q} \frac{\log(p)}{p^s} = \sum_p \left(\frac{1}{q-1} \sum_{\chi \in \hat{G}_q} \overline{\chi(a)}\,\chi(p) \right) \frac{\log(p)}{p^s}$$

absolutely convergent for $\mathrm{Re}(s) > 1$, so switch order

$$\frac{1}{q-1} \sum_{\chi \in \hat{G}_q} \overline{\chi(a)} \left(\sum_p \frac{\chi(p)\log(p)}{p^s} \right)$$

As $s \to 1^+$, this is $\sim \frac{1}{s-1}$ if $\chi = \chi_0$ (easy)

bounded if $\chi \neq \chi_0$ (hard)

Lecture 4 (2011-02-04)

$$\sum_{p \equiv a \bmod q} \frac{\log(p)}{p^s} = \sum_p \begin{pmatrix} 1 \text{ if } p \equiv a \bmod q \\ 0 \text{ otherwise} \end{pmatrix} \frac{\log(p)}{p^s} =$$

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$G_q = (\mathbb{Z}/q\mathbb{Z})^\times$ $\qquad \sum_p \frac{1}{q-1} \sum_{\chi \in \widehat{G}_q} \overline{\chi(a)} \chi(p) \frac{\log(p)}{p^s} =$

$$\frac{1}{q-1} \sum_{\chi \in \widehat{G}_q} \bar{\chi}(a) \sum_p \frac{\chi(p)\log(p)}{p^s}$$

L-Series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

If $\chi = \chi_0 \in \widehat{G}_q$, $L(s,\chi_0)$ is almost $\zeta(s)$, but

$\chi_0 : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

$\chi_0(a) = 1$ for all a

but $\chi_0(b) = 0$ if $b \notin (\mathbb{Z}/q\mathbb{Z})^\times$

So $L(s,\chi_0) = \underbrace{\left(1 - \frac{1}{q^s}\right)}_{\text{bounded from 0 if } \mathrm{Re}(s) > 0} \zeta(s)$

$$\frac{d}{ds} \log(L(s,\chi)) = \frac{1}{ds}\left(\sum_p -\log\left(1 - \frac{\chi(p)}{p^s}\right)\right) \qquad \mathrm{Re}(s) > 1$$

$$\frac{d}{ds}\left(\sum_p \sum_{u=1}^{\infty} \frac{1}{u}\left(\frac{\chi(p)}{p^s}\right)^u\right) = \sum_{u=1}^{\infty} \sum_p \frac{1}{u} \frac{\chi(p)^u (-u\log(p))}{p^{us}} =$$

$$-\sum_p \frac{\chi(p)\log(p)}{p^s} - \left(\sum_{u \geq 2} \sum_p \frac{\chi(p)^u \log(p)}{p^{us}}\right)$$

HW – analytic for $\mathrm{Re}(s) > \frac{1}{2}$

$$\operatorname*{Res}_{s=1}\left(\sum_{p\equiv a \bmod q}\frac{\log(p)}{p^s}\right)=\frac{1}{q-1}\sum_{\chi\in\widehat{G_q}}\overline{\chi}(a)\operatorname*{Res}_{s=1}\left(\sum_p\frac{\chi(p)\log(p)}{p^s}\right)=$$

$$\frac{1}{q-1}\sum_{\chi\in\widehat{G_q}}\overline{\chi}(a)\operatorname*{Res}_{s=1}\left(-\frac{d}{ds}(\log(L(s,\chi)))\right)$$

↑ equal because remainder term analytic for $\mathrm{Re}(s)>1$

$$\operatorname*{Res}_{s=a}\frac{d}{ds}\log(F(s))=\operatorname*{Res}_{s=a}\frac{F'(s)}{F(s)}=\operatorname*{ord}_{s=a}F(s)$$

$$F(s)=(s-a)^e G(s),\quad G(a)\neq 0,\infty$$

$$\frac{F'(s)}{F(s)}=\frac{e}{s-a}+\frac{G'}{G}$$

$$\zeta(s)-\frac{1}{s-1}=\sum_{n=1}^{\infty}\frac{1}{n^s}-\int_1^{\infty}\frac{dx}{x^s}\qquad \mathrm{Re}(s)>1$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^s}-\sum_{n=1}^{\infty}\int_n^{n+1}\frac{dx}{x^s}=\sum_{n=1}^{\infty}\int_0^1\frac{1}{n^s}dx-\sum_{n=1}^{\infty}\int_0^1\frac{dx}{(x+n)^s}$$

$$=\int_0^1\sum_{n=1}^{\infty}\left(\frac{1}{n^s}-\frac{1}{(n+x)^s}\right)dx$$

$$\frac{1}{n^s}\left(1-\frac{1}{(1+\frac{x}{n})^s}\right)\qquad \sum_{n=1}^{\infty}\int_0^1\frac{sx}{n^{s+1}}dx\approx s\,c\,\zeta(s+1)$$

HW: Try to figure out $\lim_{s\to 1}\zeta(s)-\frac{1}{s-1}$ using

$\lim_{x\to\infty}\sum_{n\le x}\frac{1}{n}-\log(x)=\gamma$ (hint: maybe answer is $2\gamma-1$?)

$$\operatorname*{Res}_{s=1} \sum_{p \equiv a \bmod q} \frac{\log(p)}{p^s} = \frac{1}{q-1} - \frac{1}{q-1} \sum_{\substack{\chi \in \widehat{G_q} \\ \chi \neq \chi_0}} \overline{\chi}(a) \operatorname{ord}_{s=1}(L(s,\chi))$$

Lemma. Let $a_i \in \mathbb{C}$ be such that $\sum_{n=1}^{N} a_n$ is bounded independent of N

Then $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $\operatorname{Re}(s) > 1$

Pf. $b_N = \sum_{i=1}^{N} a_i$, so $a_n = b_n - b_{n-1}$

Thus $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n - b_{n-1}}{n^s}$ $\operatorname{Re}(s) > 1$

$$= \sum_{n=1}^{\infty} \frac{b_n}{n^s} - \sum_{n=1}^{\infty} \frac{b_n}{(n+1)^s} = \sum_{n=1}^{\infty} \left| b_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right|$$

(b_n bounded; $\frac{1}{n^s}\left(1 - \left(1+\frac{1}{n}\right)^{-s}\right)$)

$$< C \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \frac{-s}{n} \right|, \text{ converges for } \operatorname{Re}(s) > 0$$

$$\sum_{n=1}^{N} \chi(n) = \sum_{m=0}^{\lfloor \frac{N}{q} \rfloor} \underbrace{\sum_{r=0}^{q-1} \chi(mq+r)}_{0} + \underbrace{\sum_{r=0}^{N - q\lfloor \frac{N}{q} \rfloor} \chi(r)}_{\text{bounded}}$$

Hence $L(s,\chi)$ analytic for $\operatorname{Re}(s) > 0$, $\chi \neq \chi_0$

Remains to show: $L(1,\chi) \neq 0$,

Prop. $F(s) = \prod_{\chi \in \widehat{G}_q} L(s,\chi)$. Let $s \in \mathbb{R}$, $s > 1$

$F(s) \in \mathbb{R}$ and $F(s) > 1$.

Pf. $\log(F(s)) = \sum_{\chi} \sum_{p} \sum_{u=1}^{\infty} \frac{\chi(p)^u}{u p^{us}} =$

$$\sum_{u=1}^{\infty} \frac{1}{u} \sum_{p} \frac{1}{p^{us}} \sum_{\chi \in \widehat{G}_q} \chi(p^u) = \sum_{u=1}^{\infty} \frac{1}{u} \sum_{\substack{p \\ p^u \equiv 1 \bmod q}} \frac{1}{p^{us}} \cdot (q-1)$$

≥ 0

Since $p^{q-1} \equiv 1 \bmod q$

Prop. $\chi \in \widehat{G}_q$ and assume $\chi((\mathbb{Z}/q\mathbb{Z})^\times) \neq \mathbb{R}$.

Then $L(1,\chi) \neq 0$.

Pf. Suppose $L(1,\chi) = 0$.

$1 \leq F(s) = L(s,\chi_0)\, L(s,\chi)\, L(s,\bar{\chi}) \prod_{\text{other } \chi} L(s,\chi)$

as $s \to 1^+$ — pole at $s=1$ — zero at $s=1$ — zero at $s=1$ — all analytic at $s=1$

$L(s,\bar{\chi}) = \overline{L(s,\chi)}$

Contradiction

Lecture 5 (2011-02-07)

Last time: $\sum_{p \equiv a \bmod q} \frac{\log(p)}{p^s}$ has a pole at $s=1$ 2011-2-7 (5)

provided $L(1,\chi) \neq 0$ for all $\chi \in \widehat{G}$, $\chi \neq \chi_0$

Done if χ is complex.

Lemma. Let $\chi \in \widehat{G}$ be a real character, $\chi \neq \chi_0$. Then $L(1,\chi) \neq 0$
(If you think about it, you see χ must be $(\frac{\cdot}{q})$)

Pf. Assume $L(1,\chi) = 0$. Look at

$$\psi(s) = \frac{L(s,\chi_0)\,L(s,\chi)}{L(2s,\chi_0)}$$

As $s \to 1^+$, $\underbrace{L(s,\chi_0)}_{\text{simple pole}}\,\underbrace{L(s,\chi)}_{0}$ ← analytic at $s=1$

$L(2s,\chi_0)$ analytic for $\mathrm{Re}(s) > \frac{1}{2}$, pole at $s = \frac{1}{2}$

So $\psi(s)$ analytic for $\mathrm{Re}(s) \geq \frac{1}{2}$, $\psi(\frac{1}{2}) = 0$.

$$\psi(s) = \prod_p \frac{(1-\chi_0(p)p^{-s})^{-1}(1-\chi(p)p^{-s})^{-1}}{(1-\chi_0(p)p^{-2s})^{-1}} =$$

$$\prod_{p \neq q} \frac{1-p^{-2s}}{(1-p^{-s})(1-\chi(p)p^{-s})} =$$

$$\prod_{\substack{p \\ \chi(p)=1}} \frac{1+p^{-s}}{1-p^{-s}} = \prod_{\chi(p)=1} (1+p^{-s})(1+p^{-s}+p^{-2s}+\dots)$$

$$\prod_{\chi(p)=1} (1+2p^{-s}+2p^{-2s}+2p^{-3s}+\dots) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad a_n \geq 0 \quad a_1 = 1$$

So $\gamma(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \geq 0$ and $a_1 = 1$, $\lim_{s \to \frac{1}{2}} \gamma(s) = 0$.

Look at Taylor series around $s = 2$ (Silverman: don't know what motivates $s=2$)

$$\gamma(s) = \sum_{m=0}^{\infty} b_m (s-2)^m$$

$$b_m = \frac{1}{m!} \frac{d^m}{ds^m} \gamma(s) \Big|_{s=2} = \frac{1}{m!} \sum_{n=1}^{\infty} \frac{(-\log(n))^m a_n}{n^2}$$

Let $b_m = (-1)^m c_m$, with $c_m \geq 0$.

$$c_0 = b_0 = \gamma(2) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} \geq 1 \quad \text{since } a_1 = 1$$

$$\gamma(s) = \sum_{m=0}^{\infty} (-1)^m c_m (s-2)^m = \sum_{m=0}^{\infty} c_m (2-s)^m$$

Let $s \to \frac{1}{2}$

$$\gamma\left(\tfrac{1}{2}\right) = \underbrace{c_0}_{\geq 1} + \underbrace{\sum_{m=1}^{\infty} c_m \left(\tfrac{3}{2}\right)^m}_{\geq 0} \geq 1, \text{ contradiction}$$

Let $\chi_q = \left(\frac{\cdot}{q}\right)$. We showed $L(1, \chi_q) \neq 0$.

What is $\sup\{\varepsilon > 0 \mid L(s, \chi_q) \neq 0 \text{ for all } 1 - \varepsilon < s \leq 1\}$?

Best so far: $\varepsilon = \frac{\text{constant}}{\sqrt{q}}$

Helps in determining $M_{a,q} = \min\{p \mid p \equiv a \bmod q\}$

Back to class field theory!

Motivating Question: Fix $K/\mathbb{Q}$. If L/K is unramified, how big can $[L:K]$ be?

Answer: Arbitrarily large (Golod–Shafarevich). $\exists K$ such that it can be

For example, $K = \mathbb{Q}(\sqrt{-2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13})$.

This is different if L/K is required to be abelian

Thm*. Fix $K/\mathbb{Q}$. There is a finite extension L/K that is the maximal abelian and unramified extension of K, and $[L:K]$ is finite.

L is called the Hilbert class field of K.

Prop. L/K abelian, $\mathfrak{p}$ unramified

a) $(\mathfrak{p}, L/K)$ has order $f(\mathfrak{p})$ in $G_{L/K}$

b) $\mathfrak{p}$ splits completely in $L \iff (\mathfrak{p}, L/K) = 1$

c) Let $J \subset \mathcal{O}_L$ be unramified (i.e. primes dividing J are unramified). Then $(N_{L/K}(J), L/K) = 1$.

(This implies $I_K \xrightarrow{(\cdot, L/K)} G_{L/K}$, $I_K \to I_K / N_{L/K}(I_L)$)

a) $(\mathfrak{P}, L/K)$ generates $D(\mathfrak{P}) \xrightarrow{\sim} \underbrace{Gal(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_{\mathfrak{p}})}_{\text{cyclic of order } f(\mathfrak{P})}$

b) $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1 \cdots \mathfrak{P}_{r(\mathfrak{p})}$, $r(\mathfrak{p}) f(\mathfrak{p}) = [L:K]$

$\mathfrak{p}$ splits completely $\Leftrightarrow f(\mathfrak{p}) = 1 \Leftrightarrow (\mathfrak{p}, L/K)$ has order 1

c) $J = \prod \mathfrak{P}_i$

$(N_{L/K} J, L/K) = \prod (N_{L/K} \mathfrak{P}_i, L/K) = \prod (\mathfrak{p}_i^{f(\mathfrak{P}_i)}, L/K)$

where $\mathfrak{p}_i = N_{L/K}(\mathfrak{P}_i)$

$= \prod (\mathfrak{p}_i, L/K)^{f(\mathfrak{P}_i)} = 1$

A side on power-residue symbols

$\left(\frac{a}{p}\right)_2 \in \{\pm 1\}$, $\left(\frac{a}{p}\right)_2 \equiv a^{\frac{p-1}{2}} \bmod p$

So we work in $\mathbb{Q}(\zeta_n)$, $\mathbb{Z}[\zeta_n]$

Let $\mathfrak{P} \in \mathrm{Spec}(\mathbb{Z}[\zeta_n])$, $\mathfrak{P} \nmid n$, $\alpha \in \mathbb{Z}[\zeta_n]$, $\mathfrak{P} \nmid \alpha$

Def. $\left(\frac{\alpha}{\mathfrak{P}}\right)_n \in \mu_n$, unique $\left(\frac{\alpha}{\mathfrak{P}}\right) \equiv \alpha^{\frac{N(\mathfrak{P})-1}{n}} \bmod \mathfrak{P}$

$\left(\frac{\alpha}{\mathfrak{P}}\right) = 1 \Leftrightarrow \alpha \equiv \beta^n \bmod \mathfrak{P}$

Example. $K = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$, $L = K(\sqrt[3]{2})$

$G_{L/K} \cong \mathbb{Z}/3\mathbb{Z}$

$\sigma \longleftrightarrow i(\sigma)$ $\qquad \sigma(\sqrt[3]{2}) = \zeta_3^{i(\sigma)}\sqrt[3]{2}$

Let π be a prime in $\mathbb{Z}[\zeta_3]$, $\pi \nmid 6$.

$$\left(\frac{\pi, L/K}{}\right)(\sqrt[3]{2}) \equiv (\sqrt[3]{2})^{N(\pi)} \bmod \pi$$

$$\equiv 2^{\frac{N(\pi)-1}{3}} \cdot \sqrt[3]{2} \bmod \pi$$

$$\equiv \left(\frac{2}{\pi}\right)_3 \sqrt[3]{2} \bmod \pi$$

action of Artin symbol is reciprocity symbol

Lecture 6 (2011-02-09)

L/K abelian and unramified (at all places, including infinite places) 2011-2-9 (6)

so $(L/K, \mathfrak{a})$ is defined for all $\mathfrak{a} \in I_K$

Thm*. Let L/K be the largest abelian unramified extension of K. (L is called the Hilbert class field of K).

a) $(L/K, \cdot) : I_K \xrightarrow{\text{onto}} G_{L/K}$

b) kernel is P_K

hence $Cl_K \xrightarrow{\sim} G_{L/K}$

$$\{\text{abelian unramified } M/K\} \longleftrightarrow \{\text{subgroups of } Cl_K\}$$

Corollary. $\mathcal{O}_K$ is a PID $\iff$ K has no abelian unramified extensions

$$\{\text{infinite primes of } K\} \longleftrightarrow \{K \hookrightarrow \mathbb{R}\} \cup \{K \hookrightarrow \mathbb{C}, K \not\subset \mathbb{R}\}/\sim$$

($\sim$: complex conjugation)

Complex primes never ramify

real primes ramify if there is a complex place of L lying over p

ramification index!

$$r_1 + 2r_2 = n$$

$\mathbb{Q}(i)$ p_∞ but there are really two,

$|$ $|$ hence ramification

$\mathbb{Q}$ p_∞ index is 2

1 and 2 are only possible indexes because only extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$

Def. A modulus is a formal product

$$\mathfrak{m} = \prod_{\mathfrak{p} \text{ places}} \mathfrak{p}^{n_\mathfrak{p}} \quad \text{with} \quad n_\mathfrak{p} \geq 0, \; n_\mathfrak{p} > 0 \text{ for only finitely many } \mathfrak{p}$$

$n_\mathfrak{p} = 0$ if $\mathfrak{p}$ is complex

$n_\mathfrak{p} = 0$ or 1 if $\mathfrak{p}$ is real

$$\mathfrak{m} = \mathfrak{m}_0 \mathfrak{m}_\infty$$

finite primes, infinite primes

$$I_K(\mathfrak{m}) = \{ \mathfrak{a} \in I_K \mid \gcd(\mathfrak{a}, \mathfrak{m}_0) = 1 \}$$

$$P_{K,1}(\mathfrak{m}) = \{ (\alpha) \mathcal{O}_K \in P_K \mid \alpha \equiv 1 \bmod \mathfrak{m}_0 \text{ and } \sigma_\mathfrak{p}(\alpha) > 0 \text{ for all infinite } \mathfrak{p} \mid \mathfrak{m} \}$$

$\sigma_\mathfrak{p} : K \hookrightarrow \mathbb{R}$

Example. $K = \mathbb{Q}$, $m \in \mathbb{Z}$

$$P_{\mathbb{Q},1}(m) = \{ (a) \mathbb{Z} \mid a \equiv 1 \bmod m \}$$

$$P_{\mathbb{Q},1}(m \cdot \infty) = \{ (a) \mathbb{Z} \mid a \equiv 1 \bmod m, \; a > 0 \}$$

When $\mathfrak{m} = 1$, $I_K(\mathfrak{m}) = I_K$ $P_{K,1}(\mathfrak{m}) = P_K$

Life would be so much easier if there were no real fields, — but it goes on.

Thm: $I_K(\mathfrak{m})/P_{K,1}(\mathfrak{m})$ is finite

Pf Sketch, $(I_K(\mathfrak{m}) : P_K \cap I_K(\mathfrak{m}))$ finite using result for classical I_K, P_K

Now show $(P_K \cap I_K(\mathfrak{m}) : P_{K,1}(\mathfrak{m}))$ also finite

Def, $H \subset I_K(\mathfrak{m})$ is called a congruence subgroup if $H \supset P_{K,1}(\mathfrak{m})$. Then $I_K(\mathfrak{m})/H$ is called a generalized ideal class group.

ORP: Assume $\mathfrak{m}$ is divisible by all primes (including infinite primes) that ramify in L/K

If ORP, $\left(\frac{L/K}{\cdot}\right) : I_K(\mathfrak{m}) \longrightarrow G_{L/K}$ is well-defined

$\Phi_{\mathfrak{m}}$, or $\Phi_{L/K,\mathfrak{m}}$

Artin Reciprocity Thm

L/K abelian

a) If $\mathfrak{m}$ is a modulus ORP, then $\Phi_{\mathfrak{m}} : I_K(\mathfrak{m}) \to G_{L/K}$ is surjective

b) $\exists\, \mathfrak{m}$, ORP, such that $\ker(\Phi_{\mathfrak{m}})$ is a congruence subgroup for $\mathfrak{m}$

c) $\exists \mathfrak{m}$ satisfying $\alpha \mathcal{O}_K \in P_{K,1}(\mathfrak{m}) \Rightarrow \Phi_{\mathfrak{m}}(\alpha \mathcal{O}_K) = 1$

$\left(\alpha \equiv 1 \bmod^{*} \mathfrak{m}\right)$ $\qquad$ $\prod \Phi_{\mathfrak{m}}(P_i)$

Ex. $L = \mathbb{Q}(\zeta_n)$, $K = \mathbb{Q}$ $M = n \cdot \infty$

$\Phi_m : I_{\mathbb{Q}}(n \cdot \infty) \to Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

$\sigma \leftrightarrow i(\sigma)$

$\sigma(\zeta_n) = \zeta_n^{i(\sigma)}$

$\Phi_m(a\mathbb{Z})(\zeta_n) = \zeta_n^{|a|}$

$a\mathbb{Z} \in P_{\mathbb{Q},1}(n \cdot \infty) \Leftrightarrow a \equiv 1 \bmod n,\ a > 0$

$\ker(\Phi_m) \supseteq P_{\mathbb{Q},1}(n \cdot \infty)$ (actually in this case equal)

Note: $M \mid N \Rightarrow P_{K,1}(N) \subseteq P_{K,1}(M)$

Same theory for $K = \mathbb{Q}(\sqrt{-D})$, $D > 0$ using elliptic curves

Lecture 7 (2011-02-11)

Artin Reciprocity L/K abelian 2011-2-11 (7)

a) $I_K(M) \xrightarrow{\phi_{L/K,M}} G_{L/K}$ is surjective for M a DRP

b) $\exists M$ such that $\ker(\phi_{L/K,M}) \supset P_{K,1}(M)$

Conductor Theorem

$\exists!$ modulus $\mathfrak{f} = \mathfrak{f}(L/K)$ DRP of L/K, such that

a) $\mathfrak{p} \mid \mathfrak{f}(L/K) \iff \mathfrak{p}$ ramifies in L/K

b) If M is DRP for L/K and $\ker(\phi_{L/K,M}) \supset P_{K,1}(M)$

then $\mathfrak{f}(L/K) \mid M$.

$\mathfrak{f}(L/K)$ = conductor of L/K

HW: Show $\mathfrak{f}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \begin{cases} 1 & \text{if } n \le 2 \\ n/2\,\infty & \text{if } n \ge 3,\ 2 \mid n,\ 4 \nmid n \\ n\,\infty & \text{otherwise} \end{cases}$

Existence Theorem

Let M be a modulus for K, let $S \subset I_K(M)$ be a subgroup with $S \supset P_{K,1}(M)$. Then $\exists!$ abelian L/K with ramified primes dividing M satisfying

$\ker(\phi_{L/K,M}) = S$, or equivalently

$\phi_{L/K,M} : I_K(M)/S \xrightarrow{\sim} G_{L/K}$

Cor. L/K, F/K abelian extensions (in some algebraic closure). TFAE:

a) $L \subset F$

b) $\exists$ modulus $\mathfrak{m}$ DRP in L and F such that
$$\ker(\Phi_{L/K,\mathfrak{m}}) \supset \ker(\Phi_{F/K,\mathfrak{m}}) \supset P_{K,1}(\mathfrak{m})$$

Pf. a) $\Rightarrow$ b)

Let $\mathfrak{m}$ be a modulus such that
$$\ker(\Phi_{L/K}) \supset P_{K,1}(\mathfrak{m})$$
$$\ker(\Phi_{F/K}) \supset P_{K,1}(\mathfrak{m})$$

$$\begin{array}{ccc} I_K(\mathfrak{m}) & \xrightarrow{\phi_{F/K}} & G_{F/K} \\ \| & & \downarrow \sigma \mapsto \sigma|_L \\ I_K(\mathfrak{m}) & \xrightarrow{\Phi_{L/K}} & G_{L/K} \end{array}$$

enough to check on primes, which generate $\cdot$ $(\mathfrak{p}, F/K)|_L = (\mathfrak{p}, L/K)$ because

$$(\mathfrak{p}, F/K)(\alpha) \equiv \alpha^{N(\mathfrak{p})} \bmod \mathfrak{P}_F \Rightarrow (\mathfrak{p}, F/K)(\alpha)^{k'} \equiv \alpha^{N(\mathfrak{p})} \bmod \mathfrak{P}_L$$

b) $\Rightarrow$ a) $\phi_{F/K} : I_K(\mathfrak{m}) \to G_{F/K}$

$\phi_{F/K}(\ker(\Phi_{L/K})) = H \subset G_{F/K}$, and let $\tilde{L} = F^H$ fixed field of H

So $\tilde{L} \subset \mathfrak{m}$, and using a) $\Rightarrow$ b), we get

$$I_K(\mathfrak{m}) \xrightarrow{\phi_{\tilde{L}/K}} G_{\tilde{L}/K} \qquad \phi^{-1} \text{ — erase}$$

Kronecker-Weber

Thm. Let $L/\mathbb{Q}$ be abelian. Then $\exists\, n \in \mathbb{N}$ such that $L \subset \mathbb{Q}(\zeta_n)$

Pf. By Artin reciprocity, L has a modulus M with $\ker(\Phi_{L/\mathbb{Q}}) \supset P_{\mathbb{Q},1}(M)$

WLOG, write $M = n \cdot \infty$. But $\ker(\Phi_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}) = P_{\mathbb{Q},1}(n \cdot \infty)$

Hence $\ker(\Phi_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}) \subset \ker(\Phi_{L/\mathbb{Q}})$

Hence $\mathbb{Q}(\zeta_n) \supseteq L$.

In particular, $\forall\, p \in \mathbb{Z}$, $\exists\, n$ with $\mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(\zeta_n)$.

Suffices to do $\mathbb{Q}(\sqrt{\pm p})$ p prime

$$\mathrm{Disc}(x^p - 1) = \prod_{\substack{\text{roots of} \\ x^p - 1}} p x^{p-1} = p^p \cdot \text{roots of unity}$$

$$= \prod_{\substack{0 \le i,j < p \\ i \neq j}} (\zeta_p^i - \zeta_p^j) = \left(\pm \prod_{0 \le i < j < p} (\zeta_p^i - \zeta_p^j) \right)^2 \in \mathbb{Q}(\zeta_p)$$

So $\exists\, A \in \mathbb{Q}(\zeta_p)$ with $A^2 = \pm p \left(p^{\frac{p-1}{2}}\right)^2$

Hence $\left(\frac{A}{p^{\frac{p-1}{2}}}\right)^2 = \pm p$, since $\sqrt{\pm p} \in \mathbb{Q}(\zeta_p)$,

Hence $\sqrt{p} \in \mathbb{Q}(\zeta_{4p})$

HW. $L/\mathbb{Q}$ abelian

let n = smallest integer with $L \subset \mathbb{Q}(\zeta_n)$

then $f(L/\mathbb{Q}) = \begin{cases} n & \text{if } \exists L \hookrightarrow \mathbb{R} \\ n \cdot \infty & \text{if } \nexists L \hookrightarrow \mathbb{R} \end{cases}$

Lecture 8 (2011-02-14)

Hilbert class field is the one with conductor $M=(1)$, i.e. 2011-2-14 (8)

$\exists!$ field K_H with $\ker(\phi_{K_H/H,(1)}) = P_K$

So $Cl_K = I_K/P_K \xrightarrow{\sim} G_{K_H/K}$

Prop. K_H is the maximal abelian unramified extension of K.

Pf. Let L/K be abelian unramified.

$$I_K \xrightarrow{\phi_{L/K,1}} G_{L/K}$$

$f(L/K)=(1)$, so $P_K \subset \ker(\phi_{L/K,1})$

because $\ker(\phi_{K_H/K,1}) \subseteq \ker(\phi_{L/K,1})$, we have $K_H \supseteq L$

Ray Class Field

Let M be a modulus. Then $\exists!$ abelian extension K_M/K such that $\ker(\phi_{K_M/K,M}) = P_{K,1}(M)$

$I_K(M)/P_{K,1}(M) \xrightarrow{\sim} Gal(K_M/K)$

K_M = ray class field of modulus M

HW: $K=\mathbb{Q}$, show $P_{\mathbb{Q},1}(2) = P_{\mathbb{Q},1}(2\cdot\infty)$

ray class field?

Example, $K = \mathbb{Q}$,

$M = (m) \rightsquigarrow \mathbb{Q}(\zeta_m + \zeta_m^{-1})$

$M = (m, \infty) \rightsquigarrow \mathbb{Q}(\zeta_m)$

HW, L/K abelian, show

$$\mathfrak{f}(L/K) = \gcd\{M : \ker(\Phi_{L/K}) \supseteq P_{K,1}(M)\}$$

L/K abelian, M DRP, and a congruence group for L/K, $\ker(\Phi_{L/K,M}) \supseteq P_{K,1}(M)$

$$\Phi_{L/K} : I_K(M) \to G_{L/K}$$

From before, $N_{L/K}(I_L(M)) \subseteq \ker(\Phi_{L/K,M})$

Thm: $\ker(\Phi_{L/K,M}) = N_{L/K}(I_L(M)) \cdot P_{K,1}(M)$

Local Fields

$K/\mathbb{Q}$, $\mathfrak{p}$, $\mathcal{O}_\mathfrak{p}$, $K_\mathfrak{p}$

$$\mathcal{O}_\mathfrak{p} = \varprojlim \mathcal{O}_K/\mathfrak{p}^i = \{(\alpha_0, \alpha_1, \ldots) \mid \alpha_i \in \mathcal{O}_K/\mathfrak{p}^i, \ \alpha_i \equiv \alpha_{i-1} \bmod \mathfrak{p}^i\}$$

Fact: $\mathcal{O}_\mathfrak{p}$ is an integral domain

(the completed at a prime power, it actually be isomorphic)

#1: K_p = fraction field of $\mathcal{O}_p$

#2: $v_p : K^\times \to \mathbb{Z}$

$\alpha \mathcal{O}_K = \prod \mathfrak{p}^{v_\mathfrak{p}(\alpha)}$ $\quad v_\mathfrak{p}(0) = \infty$

$|\alpha|_\mathfrak{p} = \frac{1}{p^{v_\mathfrak{p}(\alpha)}}$, $p = \mathrm{char}(\mathcal{O}_K/\mathfrak{p})$

$|\ |$ is a non-archimedean absolute value

Thm (Ostrowski) Every absolute value on $\mathbb{Q}$ comes from p-adic or usual one.

If $K_\mathfrak{p}$ = completion, $\mathcal{O}_\mathfrak{p} = \{\alpha \in K_\mathfrak{p} \mid |\alpha| \leq 1\}$
$\mathfrak{m}_\mathfrak{p} = \{\alpha \in K_\mathfrak{p} \mid |\alpha| < 1\}$

"local field" will refer to a number field completed at a non-archimedean place

Extension fields K local field, $\mathcal{O}_K = \{|x| \leq 1\}$, $\mathfrak{p}_K = \{|x| < 1\}$

$\mathfrak{p}_K \mathcal{O}_L = \mathfrak{p}_L^{e(L/K)}$, $f(L/K) = [\mathcal{O}_L/\mathfrak{p}_L : \mathcal{O}_K/\mathfrak{p}_K)$

L/K unramified, i.e. $e(L/K)=1$, then

$$\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(l_L/l_K) \cong \mathbb{Z}/f(L/K)\mathbb{Z}$$
$$x \mapsto x^{N\mathfrak{p}}$$

Thm. Unramified extensions are Galois,

$$\{L/K \text{ unramified}\} \longleftrightarrow \{\text{finite extensions } l'/l\}$$

K^{ur} = maximal unramified extension of K

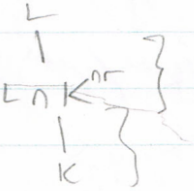
$$G(K^{ur}/K) = \varprojlim_{\rightarrow} G(L/K) \cong \varprojlim_{l'/l \text{ finite}} G(l'/l) = \widehat{\mathbb{Z}}$$

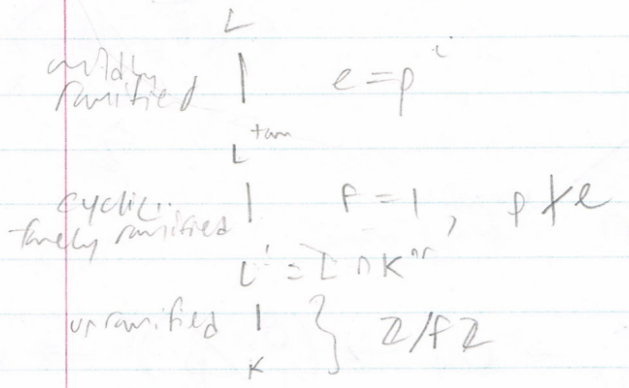
Lecture 9 (2011-02-16)

local field $K/\mathbb{Q}_p$, K^{nr} its maximal unramified extension 2011-2-16 (9)

$\mathrm{Gal}(K^{nr}/K) \cong \hat{\mathbb{Z}}$

Given L/K,





Thm, $K/\mathbb{Q}_p$, L/K abelian, then there is a natural isomorphism

$$(\cdot, L/K): K^*/N_{L/K}(L^*) \xrightarrow{\sim} G_{L/K}$$

↑ Main Thm of Local Class Field Theory

If L/K unramified, $G_{L/K} \cong \mathbb{Z}/f\mathbb{Z}$

Thm, $\{\text{finite abelian } L/K\} \leftrightarrow \{\text{open subgroups of } K^*\}$

$L \longrightarrow N_{L/K}(L^*)$

Thm. K^{ab} = max ab extension

$$1 \to O_K^\times \to K^\times \xrightarrow{v} \mathbb{Z} \to 0$$

$O_K^\times \to$ (local Artin map) $\to Gal(K^{ab}/K^{nr})$; $K^\times \to$ (local Artin map) $\to Gal(K^{ab}/K)$; $\mathbb{Z} \to$ (Frob) $\to Gal(K^{nr}/K)$

$$1 \to Gal(K^{ab}/K^{nr}) \to Gal(K^{ab}/K) \to Gal(K^{nr}/K) \to 0$$

$$Gal(K^{nr}/K) \cong Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}, \quad x \mapsto x^{Np}$$

Def of $(\cdot, L/K)$ in global:

via global class field theory — group cohomology — Lubin-Tate formal groups

$$1 \to O_{K,1}^\times \to O_K^\times \to (O_K/p)^\times \to 1$$

$O_{K,1}^\times = \{\alpha \in O_K^\times \mid \alpha \equiv 1 \bmod p\}$

$(O_K/p)^\times$: cyclic of order prime to p

$\cong \mathbb{Z}/(Np-1)\mathbb{Z}$

$$\begin{array}{ccccccccc} 1 & \to & O_{K,1}^\times & \to & O_K^\times & \to & (O_K/p)^\times & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & Gal(K^{ab}/K^{tame}) & \to & Gal(K^{ab}/K^{nr}) & \to & Gal(K^{tame}/K^{nr}) & \to & 1 \end{array}$$

So what does $O_{K,1}^\times$ look like? Note $\mathbb{R}^\times \cong \mathbb{R}^+ \times \ldots$ by log

"p-adic Lie group"

$O_{K,1}^\times \cong p^+$ (divisible by p, less small)

$$\alpha \mapsto \log(\alpha) = \log(1 + (\alpha - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\alpha-1)^n}{n!}$$

Lemma $K/\mathbb{Q}_p$, $a_n, b_n \in \mathcal{O}_K$, i.e. $|a_n|, |b_n| \leq 1$

a) $\sum a_n \frac{x^n}{n}$ converges $\forall\, |x| < 1$, i.e. $v(x) > 0$

b) $\sum b_n \frac{x^n}{n!}$ converges $\forall$, $v(x) > \frac{e(K/\mathbb{Q}_p)}{p-1}$

If, Converges $\Leftrightarrow$ n^{th} term $\to 0$ $\Leftrightarrow$ $v(n^{th}\text{ term}) \to \infty$

a) $v\left(a_n \frac{x^n}{n}\right) = n v(x) - v(n) \geq n v(x) -$

b) $v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor < \frac{n}{p-1}$ $\qquad e(K/\mathbb{Q}_p)$

$v(x) > \frac{v(p)}{p-1}$

Prop Let $r > \frac{e(K/\mathbb{Q}_p)}{p-1}$. Then there are isomorphisms

$$(\mathfrak{p}^r, +) \rightleftarrows (1+\mathfrak{p}^r, \cdot)$$

$$x \longmapsto \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(y-1)^n}{n} \longleftarrow y$$

Corollary: $\mathcal{O}_{K,1}^{\times}$ has a subgroup of finite index that is isomorphic to $(\mathfrak{p}^r, +)$

$$(\mathfrak{p}, +) \hookrightarrow (1+\mathfrak{p}\mathbb{Z}_p, \cdot)$$

Lecture 10 (2011-02-18)

Adeles + Ideles

2011-2-18

(10)

$K/\mathbb{Q}$ number field

$M_K = M_K^\circ \cup M_K^\infty$

absolute values on K

M_K° = non archimedian = primes $\mathfrak{p}$

M_K^∞ = $K \hookrightarrow \mathbb{R}$ or pairs of $K \hookrightarrow \mathbb{C}$

$$\mathcal{O}_K \hookrightarrow \prod_{v \in M_K^\infty} K_v \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

$$\mathcal{O}_{K,S} \hookrightarrow \prod_{v \in S} K_v$$ ⇝ suggests $K \hookrightarrow \prod_{v \in M_K} K_v$

Ring of adeles of $K = \mathbb{A}_K$

$$\mathbb{A}_K = \left\{ \alpha = (\alpha_v)_{v \in M_K} \in \prod_{v \in M_K} K_v \;\middle|\; \alpha_v \in \mathcal{O}_v \text{ for all but finitely many } v \right\}$$

(define $\mathcal{O}_v = K_v$ if $v \in M_K^\infty$)

define ideles $\mathbb{I}_K = \mathbb{A}_K^\times$

$$\mathbb{I}_K = \left\{ \alpha \in \prod_{v \in M_K} K_v^\times \;\middle|\; \alpha_v \in \mathcal{O}_v^\times \text{ for all but finitely many } v \right\}$$

Note that we get $K \hookrightarrow \mathbb{A}_K$

$K^\times \hookrightarrow \mathbb{I}_K$

If $(X_i)_{i \in I}$ are topological spaces,

$X = \prod X_i$ is a topological space, base of open sets is $\prod_{i \in J} U_i \times \prod_{i \in I \setminus J} X_i$

Typical open set in $\mathbb{A}_K$

$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v$ (S finite, U_v open in K_v)

topology on $\mathbb{I}_K \neq$ subspace topology from $\mathbb{A}_K$

In general, if R is a topological ring, then $R^\times$ is a topological group, with topology induced by

$$R^\times \hookrightarrow R \times R$$
$$a \longmapsto (a, a^{-1})$$

Distances in $\mathbb{I}_K$: $\alpha = (\alpha_v) \in \mathbb{I}_K$, define

$$\|\alpha\| = \prod_{v \in M_K} |\alpha_v|_v$$

Prop. $K \hookrightarrow \mathbb{A}_K$, $K^\times \hookrightarrow \mathbb{A}_K^\times$ are discrete subgroups

Pf idea, Suppose $\alpha_i \in K$, $\alpha_i \to 0$ as elements of $\mathbb{A}_K$.

$\alpha_i = (\alpha_i, \alpha_i, \dots)$ $\quad 0 = (0, 0, \dots)$

For any finite $S \subset M_K$ and any $\varepsilon > 0$,

$\exists\, i$ such that $|\alpha_i|_v < \varepsilon$ for all $v \in S$

$|\alpha_i|_v \leq 1$ for all v

for example, let $S = M_K^\infty$, $\varepsilon = \frac{1}{2}$

$|\alpha_i|_v < \frac{1}{2}$, $|\alpha_i|_v \leq 1$ for $v \in M_K^0$
for $v \in M_K^\infty$

$1 = \prod_{v \in M_K} |\alpha_i|_v < \frac{1}{2}$

unless $\alpha = 0$

Refine $\mathbb{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$

$\mathbb{A}_{K,S}^\times = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$

Thm: $\mathbb{A}_K = K + \mathbb{A}_{K, M_K^\infty}$, and $\mathbb{A}_K / K$ is compact,

where $\prod_{v \in M_K^\infty} \{\alpha \in K_v \mid |\alpha|_v \leq B\} \times \prod_{v \in M_K^0} \mathcal{O}_v \longrightarrow \mathbb{A}_K / K$ (continuous surjection)

↑ finitely many open sets

$\| \ \| : \mathbb{A}_K^\times \to \mathbb{R}_{>0}^\times$ is a continuous homomorphism

$\alpha \mapsto \prod_{v \in M_K} |\alpha_v|_v$

Also have $\mathbb{A}_K^\times \twoheadrightarrow I_K$

$\alpha = (\alpha_v)_{v \in M_K} \mapsto \prod_{v \in M_K^\circ} \pi_v^{\mathrm{ord}_v(\alpha_v)}$

Denote this by $\alpha \mathcal{O}_K$. If $\alpha \in K^\times$, this notation is consistent.

The kernel of this map is

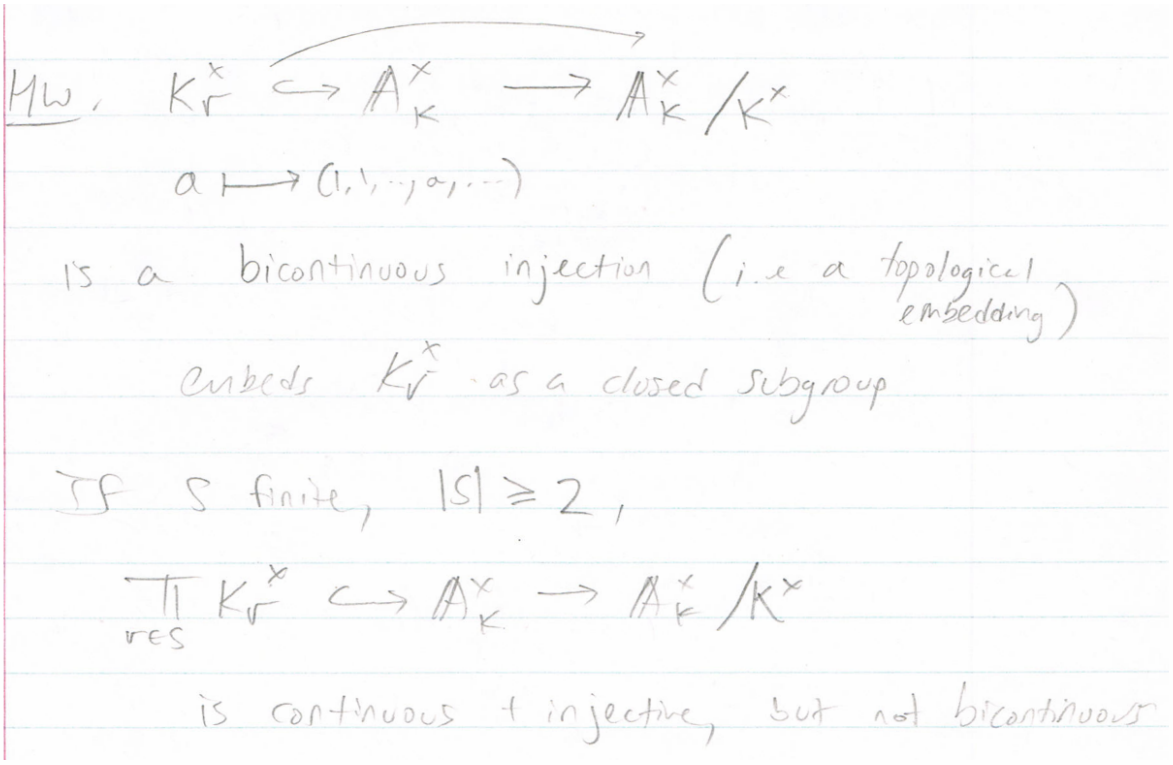
$$\prod_{v \in M_K^\infty} K_v^\times \times \prod_{v \in M_K^\circ} \mathcal{O}_v^\times = \mathbb{A}_{K, M_K^\infty}^\times$$

$\mathbb{A}_K^\times / \mathbb{A}_{K, M_K^\infty}^\times \xrightarrow{\sim} I_K$, hence

$\mathbb{A}_K^\times / K^\times \mathbb{A}_{K, M_K^\infty}^\times \xrightarrow{\sim} I_K / (K^\times) = Cl_K$

$\exists$ finite $S \subset M_K$ so that

$\mathbb{A}_K^\times = K^\times \mathbb{A}_{K,S}^\times$ – choose S so that the primes $\mathfrak{p}$ in S cover Cl_K.


Hw. $K_v^\times \hookrightarrow \mathbb{A}_K^\times \to \mathbb{A}_K^\times / K^\times$
$a \mapsto (1, 1, \ldots, a, \ldots)$
is a bicontinuous injection (i.e a topological embedding)
embeds $K_v^\times$ as a closed subgroup
If S finite, $|S| \geq 2$,
$\prod_{v \in S} K_v^\times \hookrightarrow \mathbb{A}_K^\times \to \mathbb{A}_K^\times / K^\times$
is continuous + injective, but not bicontinuous

Lecture 11 (2011-02-23)

Main Theorem of Class Field Theory 2011-2-23 (11)

$K/\mathbb{Q}$, K^{ab} = maximal abelian extension of K

Then there exists a unique continuous homomorphism

$$[\cdot, K] : \mathbb{A}_K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K) = \varprojlim_{L/K \text{ finite ab}} \mathrm{Gal}(L/K)$$

such that: (K^{ab} built out of finite abelian extensions, so enough to specify on fin)

Let L/K be a finite abelian extension.

(1) If $\alpha \in \mathbb{A}_K^\times$ satisfies $a_v \in \mathcal{O}_v$ for all v, $\alpha_v = 1$ for all v that ramify in L

Then $[\alpha, K]|_L = (\alpha \mathcal{O}_K, L/K)$

(2) If $\alpha \in K^\times$, then $[\alpha, K] = 1$.

(5) $[\cdot, K]$ is surjective

(3) Let L/K abelian extension, $\beta \in \mathbb{A}_L^\times$

$$[\beta, L]|_{K^{ab}} = [N_{L/K}(\beta), K]$$

L^{ab} not necessary K^{ab}

Def. $N_{L/K} : \mathbb{A}_L^\times \to \mathbb{A}_K^\times$, $\beta \mapsto$

(4) $K_v^\times \hookrightarrow \mathbb{A}_K^\times$

$c \mapsto (1, 1, \ldots, c, 1, \ldots)$

Let $v \in M_K$. Then

$$(N_{L/K}(\beta))_v = \prod_{\substack{w \in M_L \\ w|v}} N_{L_w/K_v}(\beta_w)$$

$[\mathcal{O}_v^\times, K]$ = inertia group of v in $\mathrm{Gal}(K^{ab}/K)$

Example: If v is unramified, $c \in K_v^\times \subset \mathbb{A}_K^\times$

$$[c, K]\big|_L = [(1,1,\dots,c,1,\dots), K]\big|_L = \mathrm{Frob}_v^{\mathrm{ord}_v(c)} \in \mathrm{Gal}(L/K)$$

Let $\mathfrak{m}$ be a modulus for K.

Def. $U_{\mathfrak{m}} = \{\alpha \in \mathbb{A}_K^\times \mid \alpha_v \in$

- Computing $[\alpha, K]\big|_L$ for some finite abelian L/K

we want to get it into the form of (1).

Given $\alpha \in \mathbb{A}_K^\times$, find some $c \in K^\times$ such that

$c\alpha_v \in \mathcal{O}_v$ for all v

Pick a modulus $\mathfrak{M}$ for L/K; for ramified primes $v \mid \mathfrak{m}$,

require $c\alpha_v \equiv 1 \bmod \mathfrak{M}\mathcal{O}_v$

(If $\mathrm{ord}_v(\alpha_v) \neq 0$, first divide α by some $a \in K^\times$ to make $\mathrm{ord}_v(a^{-1}\alpha) = 0$)

Because $[\cdot, K)$ doesn't see $K^\times$,

$$[\alpha, K]_L = [c\alpha, K]_L = (c\alpha\mathcal{O}_K, L/K)$$

Example $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_{p^n})$

$\alpha = (1, \ldots 1, \underset{\uparrow}{a}, 1, \ldots 1) \quad a \in \mathbb{Z}_p^*$

$\mathbb{Q}_p$

$(c, \ldots c, ca, c, \ldots c)$

Choose $c \in \mathbb{Z}$ with $ca \equiv 1 \bmod p^n \cdot \infty$

$c \in \mathbb{Z} \quad c > 0 \quad c \equiv a^{-1} \bmod p^n$

$$[\alpha, K]|_L = [c\alpha, K]|_L = (c\alpha\mathcal{O}_K, L/K) = \prod_{\text{prime } q \nmid c} \text{Frob}_q^{\text{ord}_q(c)}$$

$\mathcal{O}_K = \mathbb{Z}$

$$[\alpha, \mathbb{Q}](\zeta_{p^n}) = \zeta_{p^n}^{c} = \zeta_{p^n}^{a^{-1}}$$

HW $\alpha = (1, 1, \ldots \underset{\uparrow}{p^i}, 1, \ldots)$ Show $[\alpha, \mathbb{Q}]|_{\mathbb{Q}(\zeta_{p^n})} = 1$

$\mathbb{Q}_p$ spot

$\beta = (1, 1, \ldots, -p^i, \ldots)$, evaluate $[\beta, \mathbb{Q}]|_{\mathbb{Q}(\zeta_{p^n})}$.

Lecture 12 (2011-02-25)

$\mu_n \subset K$, $\alpha \in \mathcal{O}_K$, $\mathfrak{p} \nmid n\alpha$ 2011-2-25 (12)

n^{th} order residue symbol

$\left(\frac{\alpha}{\mathfrak{p}}\right)_n \in \mu_n$ such that $\alpha^{\frac{N\mathfrak{p}-1}{n}} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_n \bmod \mathfrak{p}$

If $\mathfrak{a} = \prod \mathfrak{p}_i$, $\left(\frac{\alpha}{\mathfrak{a}}\right)_n = \prod \left(\frac{\alpha}{\mathfrak{p}_i}\right)_n$

$\left(\frac{\alpha}{\cdot}\right)_n : I_K(\mathfrak{m}) \to \mu_n$ homomorphism (when $n\alpha \mid \mathfrak{m}$)

For a Kummer extension,

$$\mathrm{Gal}(K(\sqrt[n]{\alpha})/K) \hookrightarrow \mu_n$$
$$\sigma \mapsto \zeta_\sigma = \frac{\sigma(\sqrt[n]{\alpha})}{\sqrt[n]{\alpha}}$$

Prop. $\mu_n \subset K$, $L = K(\sqrt[n]{\alpha})$, $\alpha \in \mathcal{O}_K \setminus \{0\}$

Choose a modulus $\mathfrak{m}$ such that
1. Every $\mathfrak{p} \mid n\alpha$ has $\mathfrak{p} \mid \mathfrak{m}$
2. $\ker(\Phi_{L/K,\mathfrak{m}}) \supset P_{K,1}(\mathfrak{m})$, i.e. $\mathfrak{m}$ is a modulus for L/K – can always make a modulus bigger, so 1 is ok

exists by Artin reciprocity →

Then $I_K(\mathfrak{m}) \xrightarrow{\Phi_{L/K,\mathfrak{m}}} G_{L/K}$

$\left(\frac{\alpha}{\cdot}\right)_n : I_K(\mathfrak{m}) \to \mu_n$, $\sigma \mapsto \zeta_\sigma$

commutes, as .

$\left(\frac{\alpha}{\cdot}\right)_n : \frac{I_K(m)}{P_{K,1}(m)} \xrightarrow{\text{onto}} \text{Image}\left(G_{L/K} \to \mu_n\right)$

$(\mathfrak{P}, L/K)(\sqrt[n]{\alpha}) \equiv (\sqrt[n]{\alpha})^{N\mathfrak{p}} \bmod \mathfrak{P}$ $\quad \mathfrak{P}$ lies above $\mathfrak{p}$

by def of Frobenius

$\equiv \alpha^{\frac{N\mathfrak{p}-1}{n}} \sqrt[n]{\alpha} \bmod \mathfrak{P} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_n \sqrt[n]{\alpha} \bmod \mathfrak{P}$

so $\frac{(\mathfrak{P}, L/K)(\sqrt[n]{\alpha})}{\sqrt[n]{\alpha}} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_n \bmod \mathfrak{P}$

but n^{th} roots stay distinct, so $(\mathfrak{P}, L/K)(\sqrt[n]{\alpha}) = \left(\frac{\alpha}{\mathfrak{p}}\right)_n \sqrt[n]{\alpha}$

Second statement follows from the fact that $\Phi_{L/K,\mathfrak{m}}$ is surjective.

Prop. $p, q \in \mathbb{Z}$ odd primes, then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$

Pf. Let $p^* = (-1)^{\frac{p-1}{2}} p$. We know $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$,

so equivalently we want to prove $\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right)$

$\frac{I_{\mathbb{Q}}(p \cdot \infty)}{P_{\mathbb{Q},1}(p \cdot \infty)} \xrightarrow{\Phi_{\mathbb{Q}(\zeta_p)/\mathbb{Q}, p\cdot\infty}} \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$

For any subfield $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\zeta_p)$, we also get

$$I_{\mathbb{Q}}(p,\infty) / P_{\mathbb{Q},1}(p,\infty) \xrightarrow{\bar{\Phi}_{K,p,\infty}} Gal(K/\mathbb{Q})$$

$Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$, cyclic order $p-1$

Hence unique subfield $K \subseteq \mathbb{Q}(\zeta_p)$ with $[K:\mathbb{Q}] = 2$

The only (finite) prime K can be ramified at is p

Hence $K = \mathbb{Q}(\sqrt{p^*})$.

$$(\mathbb{Z}/p\mathbb{Z})^\times \xrightarrow{\sim} I_{\mathbb{Q}}(p,\infty)/P_{\mathbb{Q},1}(p,\infty) \xrightarrow[\text{onto}]{\left(\frac{p^*}{\cdot}\right)} \text{Image}\left(Gal(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}) \to \mu_2\right) \subseteq \mu_2$$

$a \bmod p \mapsto a\mathbb{Z}$

So $a \longmapsto \left(\frac{p^*}{a}\right)$ is a surjective homomorphism from

$$(\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mu_2$$

but $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic, so there is only one such.

But $a \longmapsto \left(\frac{a}{p}\right)$ is also a surjective homomorphism

so they are the same

$\left(\frac{\Pi}{\eta}\right)_n$ depends only on $\prod_{p \mid n} p$ Π and η

Group Cohomology

G a finite group, M a G-module

$\mathbb{Z}[G] = \left\{ \sum_{\sigma \in G} n_\sigma \cdot \sigma \;\middle|\; n_\sigma \in \mathbb{Z},\ n_\sigma = 0 \text{ for all but finitely many } \sigma \right\}$

group ring

$$\left(\sum n_\sigma \sigma\right)\left(\sum n_\tau \tau\right) = \sum_{\sigma,\tau} n_\sigma n_\tau \sigma\tau$$

group ring is non-commutative if G is

M is really a (left) $\mathbb{Z}[G]$-module

$$\left(\sum n_\sigma \sigma\right)(x) = \sum n_\sigma \sigma(x)$$

Examples L/K, $G = \mathrm{Gal}(L/K)$ acts on $L^+, L^\times$

$(\sigma - 1)(x) = \sigma(x) - x \qquad (\sigma - 1)(x) = \frac{\sigma(x)}{x}$

Basic object of study: $M^G = \{ x \in M \mid \sigma(x) = x \text{ for all } \sigma \in G \}$

Example G is cyclic, say $G = \langle \sigma \rangle$

$M^G = \{ x \in M \mid \sigma(x) = x \} = \ker\left((\sigma - 1) : M \to M \right)$

$(\sigma - 1)(x) = 0$

Prop. Let $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$ be an exact sequence of G-modules, where α and β are G-module morphisms (commute with G-action)

Then $0 \to M^G \to N^G \to P^G$ is exact

$\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) = M^G$ right adjoint

($\mathbb{Z}$: trivial action)

Lecture 13 (2011-02-28)

Let $L/K/\mathbb{Q}_p$ be a Galois extension of p-adic fields 2011-2-28 (13)

$$1 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{\text{ord}_L} \mathbb{Z} \to 1$$

take fixed elements

$$1 \to \mathcal{O}_K^\times \to K^\times \xrightarrow{\text{ord}_L} \mathbb{Z} \to \mathbb{Z}/e(L/K)\mathbb{Z} \to 1$$

$\| e(L/K)\,\text{ord}_L$

exact $\Leftrightarrow$ unramified

M, N G-modules, then $\text{Hom}_G(M,N) = \text{Hom}_{\mathbb{Z}}(M,N)^G$

where $\text{Hom}_{\mathbb{Z}}(M,N)$ is a G-module by $\sigma \cdot f = \sigma \circ f \circ \sigma^{-1}$

(this is equivalent to requiring $\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ M & \xrightarrow{f} & N \end{array}$ commute)

In particular $\text{Hom}_G(\mathbb{Z}, M) = M^G$, since $\sigma f(\sigma^{-1}(a)) = f(a)$ $\forall a \in M$

implies $f(1) = \sigma(f(1))$

Def A co-induced G-module is one of the form

$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ where A is an abelian group with trivial G-action

Notation $M^G = \Gamma(G, M) = H^0(G, M)$

possible to use a classifying space for group G, make this sheaf notation rigorous

Thm. There are abelian groups $H^q(G,M)$, $q \geq 0$ and, for each exact sequence $0 \to M \to N \to P \to 0$, connecting homomorphisms $H^q(G,P) \xrightarrow{\delta} H^{q+1}(G,M)$, functorial on exact sequences, such that

(1) $0 \to H^0(G,M) \to H^0(G,N) \to H^0(G,P) \xrightarrow{\delta} H^1(G,M) \to \cdots$ is exact

and (2) if M is coinduced, then $H^q(G,M) = 0$ for $q \geq 1$.

$H^q(G,M)$ is "unique".

Pf. $M \rightsquigarrow M^G = \mathrm{Hom}_G(\mathbb{Z}, M)$

Take a resolution $\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$

where F_i are free $\mathbb{Z}$-modules, and apply $\mathrm{Hom}_G(\cdot, M)$

get a complex (toss out $\mathbb{Z}$)

C^M: $0 \to \mathrm{Hom}_G(F_0, M) \to \mathrm{Hom}_G(F_1, M) \to \cdots$

$H^q(G,M) = q^{th}$ cohomology of C^M =

$$\frac{\ker\left(\mathrm{Hom}_G(F_q, M) \to \mathrm{Hom}_G(F_{q+1}, M)\right)}{\mathrm{im}\left(\mathrm{Hom}_G(F_{q-1}, M) \to \mathrm{Hom}_G(F_q, M)\right)}$$

Abstract nonsense ⇒ long exact sequence

Check $H^0(G,M) = \dfrac{\ker(\mathrm{Hom}_G(F_0,M) \to \mathrm{Hom}_G(F_1,M))}{\mathrm{Im}(0 \to \mathrm{Hom}_G(F_0,M))}$

and $\to F_1 \to F_0 \to \mathbb{Z} \to 0$ exact, so

$0 \to \mathrm{Hom}_G(\mathbb{Z},M) \to \mathrm{Hom}_G(F_0,M) \to \mathrm{Hom}_G(F_1,M)$ exact,

so $\mathrm{Hom}_G(\mathbb{Z},M) = M^G = \ker(\mathrm{Hom}_G(F_0,M) \to \mathrm{Hom}_G(F_1,M))$

so $H^0(G,M) = M^G$

(2) coinduced ⇒ trivial cohomology

Let $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$, let N be any G-module.

$\mathrm{Hom}_G(N,M) = \mathrm{Hom}_G(N, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],A))$

ϕ ↘ ↗ $(n \mapsto (\sigma \mapsto \psi(\sigma(n))))$

$\mathrm{Hom}_{\mathbb{Z}}(N,A)$ ψ

$(n \mapsto \phi(n)(e_G))$

so $\mathrm{Hom}_G(N,M) = \mathrm{Hom}_{\mathbb{Z}}(N,A)$, so on $\to F_1 \to F_0 \to \mathbb{Z} \to 0$,

since the F_i are free abelian groups, $F_i \cong_{\mathbb{Z}} \mathbb{Z}^{n_i}$,

$\mathrm{Hom}_G(F_i,M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{n_i},A) = A^{n_i}$, so the complex is exact,

so cohomology is trivial

For uniqueness: Use $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M)$

who knows?

$$0 \to M \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M) \to M' \to 0 \qquad m \mapsto (\sigma \mapsto \sigma(m))$$

$$H^q(G, M') \overset{\delta}{\cong} H^{q+1}(G, M) \quad \text{for } q \geq 1$$

because $H^q(G, \text{coinduced}) \to H^q(G, M') \to H^{q+1}(G, M) \to H^{q+1}(G, \text{coinduced})$

$H^q(G, \text{coinduced}) = 0$ and $H^{q+1}(G, \text{coinduced}) = 0$

"dimension shifting"

Def. $Z^1(G, M) = \{f : G \to M \mid f(\sigma\tau) = f(\sigma) + \sigma f(\tau)\}$

the 1-cocycles

the 1-coboundaries

$$B^1(G, M) = \{f : G \to M \mid \exists\, m \in M \text{ such that } f(\sigma) = \sigma(m) - m\}$$

check that $B^1(G, M) \subset Z^1(G, M)$

We have $H^1(G, M) = \dfrac{Z^1(G, M)}{B^1(G, M)}$.

given $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$ we get

$$0 \to M^G \to N^G \to P^G \xrightarrow{\delta} H^1(G, M) \to \cdots$$

Let $p \in P^G$, choose $n \in N$ with $\beta(n) = p$. Construct

$$G \to N$$
$$\sigma \mapsto (\sigma(n) - n)$$

$\sigma(\eta) - \eta$ is actually in M: since

$$\beta(\sigma(\eta) - \eta) = \sigma(\beta(\eta)) - \beta(\eta) = \sigma(\rho) - \rho = 0$$

so get in fact $G \to M$

$$\sigma \mapsto \alpha^{-1}(\sigma(\eta) - \eta)$$

Hilbert 90: $H^1(G_{L/K}, L^\times) = 0$

Lecture 14 (2011-03-02)

Standard complex 2011-3-2 (14)

$F_i = \mathbb{Z}[G^{i+1}]$ $(\sigma\tau_0, \ldots, \sigma\tau_i) = \sigma(\tau_0, \ldots, \tau_i)$

$\cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$

$\mathbb{Z}[G]$

$\sum n_\sigma \cdot \sigma \to \sum n_\sigma$

$F_i \cong \mathbb{Z}[G]^i$ as $\mathbb{Z}[G]^{i+1}$

$F_i \to F_{i-1}$ by $d(\tau_0, \ldots, \tau_i) = \sum_{j=0}^{i} (-1)^j (\tau_0, \ldots, \widehat{\tau_j}, \ldots \tau_i)$

HW. Prove exactness

Hint. Fix $\sigma \in G$, let $h_\sigma : F_{i-1} \to F_i$ $(\tau_0, \ldots, \tau_{i-1}) \mapsto (\sigma, \tau_0, \ldots, \tau_{i-1})$

Show $d \circ h_\sigma + h_\sigma \circ d = 1$

Can use to compute $H^i(G, M)$. Look at $f \in \mathrm{Hom}_G(F_i, M)$,

a G-module homomorphism $f : \mathbb{Z}[G^{i+1}] \to M$

Notice $f(\tau_0, \ldots, \tau_i) = \tau_0 (1, \tau_0^{-1}\tau_1, \ldots, \tau_0^{-1}\tau_i)$

f corresponds to a map $\phi : G^{i+1} \to M$ by setting

$\phi(\tau_1, \ldots, \tau_i) = f(1, \tau_1, \tau_1\tau_2, \ldots, \tau_1 \ldots \tau_i)$

$d\phi : G^{i+1} \to M$ $\quad$ ϕ is a cocycle $\Leftrightarrow d\phi = 0$

$$d\phi(\tau_1, \dots, \tau_{i+1}) = \tau_1 \phi(\tau_2, \dots, \tau_{i+1}) + \sum_{j=1}^{i} (-1)^j \phi(\tau_1, \dots, \tau_{j-1}, \tau_j \tau_{j+1}, \tau_{j+2}, \dots, \tau_{i+1}) + (-1)^{i+1} \phi(\tau_1, \dots, \tau_i)$$

1-cocycle: $\phi : G \to M$

$$0 = d\phi(\sigma, \tau) = \sigma\phi(\tau) - \phi(\sigma\tau) + \phi(\sigma)$$

i.e. $\phi(\sigma\tau) = \sigma\phi(\tau) + \phi(\sigma)$

$0 \to C \xrightarrow{\alpha} \square \underset{\gamma}{\overset{\beta}{\rightleftarrows}} A$ $\quad$ $H^2(A, C)$ classifies extensions

γ is a homomorphism iff $\gamma(x+y) - \gamma(x) - \gamma(y) = 0$

$H \subset G$ subgroup, get $H^i(G, M) \to H^i(H, M)$

$H \lhd G$, then take $(G/H)^i \to M^H$ to

$G^i \to (G/H)^i \to M^H \hookrightarrow M$

inflation – restriction

Thm. There is an exact sequence

$$0 \to H^1(G/H, M^H) \xrightarrow[i]{\inf} H^1(G, M) \xrightarrow[r]{\text{res}} H^1(H, M)$$

Pf. i injective $\quad G/H \xrightarrow{e} M^H$, call the map

$G \to G/H \xrightarrow{e} M^H \hookrightarrow M$, $\quad$ Suppose $i(c)$ is a coboundary, i.e. $\exists\, m \in M$ such that $i(c)(\sigma) = \sigma(m) - m \;\forall \sigma \in G$

Notice that this only depends on $\sigma \bmod H$

$r \circ i = 0$ because $H \to G \to G/H \xrightarrow{e} M^H \to M$ (the composite $H \to G \to M$ is $i(c)$)

$\ker(r) \subseteq \operatorname{im}(i)$ Let $c \in \ker(r)$,

$H \hookrightarrow G \xrightarrow{c} M \quad r(c) =$ coboundary

$\exists$ an $m \in M$ such that $r(c)(\sigma) = \sigma(m) - m \;\forall \sigma \in H$

Replace c by $c' : G \to M$, $c'(\sigma) = c(\sigma) - (\sigma(m) - m)$

Changed c by a $G \to M$ 1-coboundary. Now $c'(\sigma) = 0 \;\forall \sigma \in H$

Recall $G/H \to M$. Let $\gamma \in H$, $\sigma \in G$.

$\sigma H \mapsto c'(\sigma)$

$c'(\sigma\gamma) = \sigma c'(\gamma) + c'(\sigma)$ ← 1-cocycle condition

This shows $c'(\sigma H)$ is well-defined

Claim: Image $(c') \subset M^H$.

Let $\gamma \in H$, $\sigma \in G$. $\gamma \cdot c'(\sigma) = c'(\gamma\sigma) - c'(\gamma)$

$c'(H\sigma) = c'(\sigma)$

$0 \to H^1(G/H, M^H) \xrightarrow{\inf} H^1(G, M) \xrightarrow{\text{res}} H^1(H, M)^{G/H} \to$

$H^2(G/H, M^H) \to H^2(G, M)$

Spectral sequence

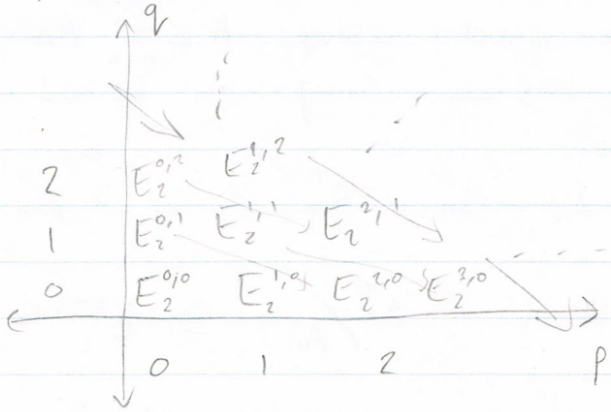
Aim: Show G/H acts on $H^1(H, M)$,

Show image of restriction is inside $H^1(H, M)^{G/H}$

Lecture 15 (2011-03-04)

Spectral Sequences 2011-3-4 (15)

$(E_2^{p,q})$



For the spectral sequences we'll be looking at, all other quadrants are 0.

$$E_2^{p,q} \xrightarrow{\delta_2^{p,q}} E_2^{p+2,q-1} \qquad \delta^{p+2,q-1} \circ \delta^{p,q} = 0$$

for level 2, we have "over 2, down 1"

define $(E_3^{p,q})$ by

$$E_3^{p,q} = \frac{\ker(\delta_2^{p,q})}{\operatorname{im}(\delta_2^{p+2,q-1})} \quad \text{and}$$

$$\delta_3^{p,q} : E_3^{p,q} \to E_3^{p+3,q-2}$$

Continuing, get $(E_r^{p,q})$ and $\delta_r^{p,q}$ for $r = 2,3,4,\dots$

For a given p,q, $E_r^{p,q}$ eventually stabilizies (when the maps into and out of $E_r^{p,q}$ land in the 0 quadrants), call this $E_\infty^{p,q}$

$r > \max(p, q+1)$

Def. The spectral sequence $(E_2^{p,q})$ converges to the graded group $A = \bigoplus_{n=0}^{\infty} A_n$ such that A_n has a composition series (as an abelian group)

$$A_n = A_n^n \supset A_n^{n-1} \supset \cdots \supset A_n^0 \supset 0$$

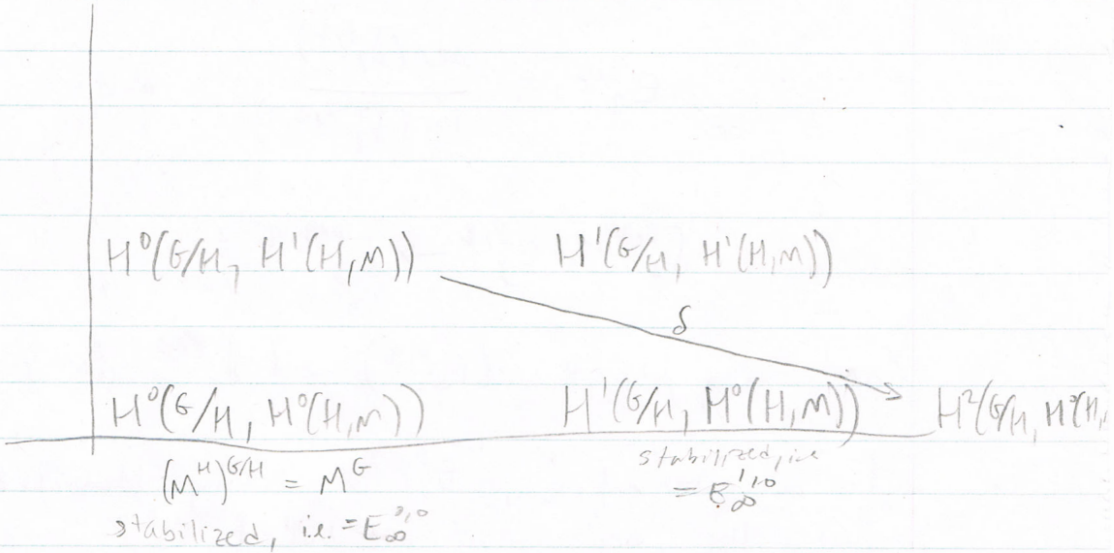
such that $A_n^q / A_n^{q-1} \cong E_2^{n-q,q}$ (?)

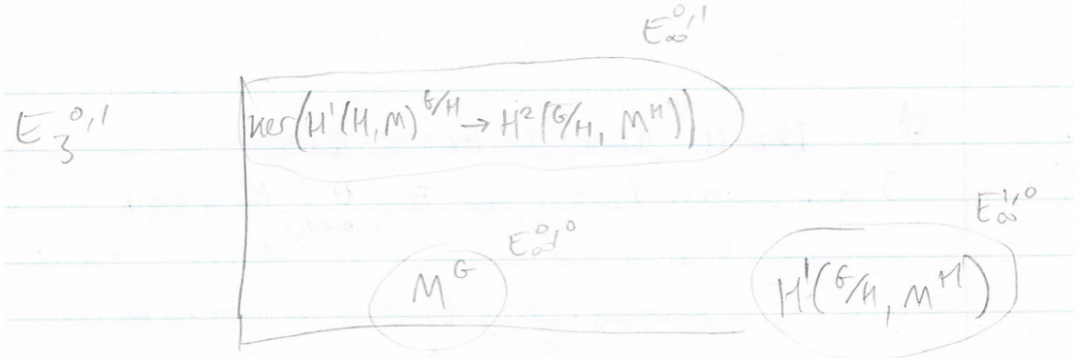
Serre–Hochschild Spectral Sequence

$$E_2^{p,q} = H^p(G/H, H^q(H,M))$$

$E_2^{p,q} \xrightarrow{\delta} E_2^{p+2,q-1}$ messy to write down

$$\boxed{H^p(G/H, H^q(H,M)) \Rightarrow \bigoplus_{n=0}^{\infty} H^n(G,M)}$$





$H^0(G,M) = E_\infty^{0,0} = M^G$ ✓

We need to have a composition series of length 2,

$$H^1(G,M) \supset A \supset 0,$$

with $H^1(G,M)/A = E_\infty^{0,1}$ $A = E_\infty^{1,0}$

$$0 \to A \to H^1(G,M) \to H^1(G,M)/A \to 0$$

$$A = E_\infty^{1,0} = H^1(G/H, M^H), \qquad H^1(G,M)/A = E_\infty^{0,1} = \ker(H^1(H,M)^{G/H} \to H^2(G/H, M^H))$$

(inf, res)

Leray spectral sequence

$X \xrightarrow{f} S$ fiber space, $f^{-1}(s) \cong F$

$$H^p(S, H^q(F,R)) \Rightarrow H^{p+q}(X)$$

Thm L/K Galois extension of fields, then
$H^q(\mathrm{Gal}(L/K), L^\times) = 0$

Pf. Recall normal basis thm.

$\exists \alpha \in L$ such that $L = \bigoplus_{\sigma \in G(L/K)} K \cdot \sigma(\alpha)$

$L^+ = \mathbb{Z}[\mathrm{Gal}(L/K)] \otimes_{\mathbb{Z}} K^+ \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathrm{Gal}(L/K)], K^+)$

So L^+ is coinduced

HW: G finite group A abelian group, trivial G-action

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$

$f \longmapsto \sum_{\sigma \in G} \sigma \otimes f(\sigma)$

Lecture 16 (2011-03-07)

Hilbert Theorem 90 L/K finite Galois, then 2011-3-7 (16)

$$H^1(G_{L/K}, L^\times) = 0$$

Lemma (Independence of characters)
Let $\chi_1, \dots, \chi_n : G \to L^\times$ be distinct characters
Then $\sum_{i=1}^{n} c_i \chi_i = 0 \Rightarrow c_1 = \cdots = c_n = 0$.

Pf. Suppose $\sum_{i=1}^{n} c_i \chi_i = 0$ is a minimal equation of dependency.

$n=1$: $c\chi = 0$, but $\chi(1) = 1$, so $c = 0$.
$n \geq 2$: Fix $\tau \in G$, Then $\forall \sigma \in G$,

$$\sum_{i=1}^{n} c_i \chi_i(\tau\sigma) = 0$$

$$c_1 \chi_1(\tau)\chi_1(\sigma) + \sum_{i=2}^{n} c_i \chi_i(\tau)\chi_i(\sigma) = 0$$

$$c_1 \chi_1(\tau)\chi_1(\sigma) + \sum_{i=2}^{n} c_i \chi_1(\tau)\chi_i(\sigma) = 0$$

$$\sum_{i=2}^{n} c_i (\chi_i(\tau) - \chi_1(\tau))\chi_i(\sigma) = 0 \quad \forall \sigma$$

other

Pf. $\sigma \in G_{L/K}$, $\sigma : L^\times \to L^\times$ let χ_σ be σ considered as a character, $\chi_\sigma(\alpha) = \sigma(\alpha)$.

Let $c : G_{L/K} \to L^\times$ be a 1-cocycle, look at
(distinct characters)

$$\sum_{\sigma \in G_{L/K}} c_\sigma \chi_\sigma$$

(non-zero since in $L^\times$)

By independence of characters, $\sum c_\sigma \chi_\sigma \neq 0$

So $\exists \alpha \in L$ such that $\beta = \sum_{\sigma \in G_{L/K}} c_\sigma \chi_\sigma(\alpha) \neq 0$

Let $\gamma \in G_{L/K}$. Compute

$$\gamma(\beta) = \sum_\sigma \gamma(c_\sigma)\,\gamma(\chi_\sigma(\alpha)) =$$

$$\sum_\sigma \frac{c_{\gamma\sigma}}{c_\gamma}\,\gamma\sigma(\alpha)$$

$$= \frac{1}{c_\gamma} \sum_\sigma c_\sigma \sigma(\alpha)$$

$$= \frac{1}{c_\gamma}\beta \qquad c_\gamma = \frac{\beta}{\gamma(\beta)} = \frac{\gamma(\beta^{-1})}{\beta^{-1}}$$

Cyclic groups

$G = \langle \sigma \mid \sigma^n = 1 \rangle \qquad T = \sigma - 1 \qquad N = \sigma^{n-1} + \cdots + \sigma + 1 \qquad \in \mathbb{Z}[G]$

$$\mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{T} \mathbb{Z}[G] \to \mathbb{Z} \to 0 \qquad TN = NT = 0$$

$\sigma \mapsto 1$ degree map

FTW! This is exact, free resolution.

$$0 \to \mathrm{Hom}_G(\mathbb{Z}[G], M) \xrightarrow{T} \cdots$$

$$H^0(G, M) = M^G \qquad H^{even}(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$$

$$H^{even} = \frac{\ker(\longrightarrow)}{\mathrm{Im}(\longrightarrow)} \qquad H^{odd}(G, \mathbb{Z}) = 0$$

Hilbert Theorem 90

L/K cyclic, $G_{L/K} = \langle \sigma \rangle$

$$0 = H^1(G_{L/K}, L^\times) = \frac{\ker(N)}{\mathrm{im}(T)}$$

Herbrand quotient

G cyclic $= \langle \sigma \rangle$

If $H^{2i}(G,M)$ and $H^{2i+1}(G,M)$ are finite, define

$$h(M) = \frac{\#H^{2i}(G,M)}{\#H^{2i+1}(G,M)}$$

Prop. $0 \to M \to N \to P \to 0$ exact. If two of $h(M), h(N), h(P)$ exist, so does the third, and $h(N) = h(M)h(P)$.

Pf. Long exact sequence

$$H^{2i}(P) \to H^{2i+1}(M) \to H^{2i+1}(N) \to H^{2i+1}(P) \to H^{2i}(M) \to H^{2i}(N) \to H^{2i}(P)$$

$$\frac{|H^{2i}(P)|\,|H^{2i+1}(N)|\,|H^{2i}(M)|}{|H^{2i+1}(M)|\,|H^{2i+1}(P)|\,|H^{2i}(N)|} = 1$$

Prop. If M is finite, then $h(M)=1$

Look at $0 \to M^G \to M \xrightarrow{T} M \to M/TM \to 0$

$0 \to H^{2i+1}(M) \to M^G/TM \xrightarrow{N} M^G \to H^{2i}(M)$

$\|$ $\omega''(N)/\mathrm{Im}(T)$ $\qquad$ $\downarrow$ M^G/NM

$|M^G| = |M/TM|$ so $|H^{2i+1}(M)| = |H^{2i}(M)| \Rightarrow h(M)=1$

Corollary $K=\mathbb{F}_q$, $L=\mathbb{F}_{q^n}$, then $N_{L/K}(L^\times)=K^\times$.

Pf. $H^1(L^\times)=0$, $G_{L/K}$ cyclic

$$h(L^*) = \frac{|H^{\mathrm{odd}}(L^\times)|}{|H^{\mathrm{even}}(L^\times)|} = 1$$

$$H^{\mathrm{even}}(L^\times) = 0 = \frac{\ker(T)}{\mathrm{Im}(N)} = \frac{K^*}{N_{L/K}(L^\times)}$$

Lecture 17 (2011-03-09)

Profinite groups 2011-3-9

(17)

$G = \varprojlim G_i$, each G_i finite

p-adic integers $\mathbb{Z}_p$, $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$, $G = \mathrm{Gal}(\bar{K}/K)$

$\mathrm{Gal}(\bar{K}/K) = \varprojlim_{L/K \text{ finite}} \mathrm{Gal}(L/K)$

basis of open sets around 1 consists of $\{\ker G \to G_i\}_{i \in I}$

Def. A topological group is topologically cyclic if there exists a dense cyclic subgroup

Examples. $\mathbb{Z}_p$, $\hat{\mathbb{Z}}$

Fundamental Theorem of Galois Theory

L/K Galois, then

$\{\text{closed subgroups } H \subset \mathrm{Gal}(L/K)\} \longleftrightarrow \{\text{intermediate fields } K \subset E \subset L\}$

$H \longmapsto L^H$

$\mathrm{Gal}(L/E) \longleftarrow E$

G profinite, M a G-module.

Prop (HW): TFAE

① $M = \bigcup_{\substack{H \lhd G \\ H \text{ open}}} M^H$

② $\forall x \in M$, $\mathrm{Stab}(x) = \{\sigma \in G \mid \sigma(x) = x\}$ is open subgroup of G

③ $G \times M \to M$ is continuous
$(\sigma, x) \mapsto \sigma(x)$

Def. M is called a discrete G-module
(hence forth dropping "discrete")

Example. $G(\bar{K}/K)$ acting on $\bar{K}$ or $\bar{K}^\times$

given $x \in \bar{K}$, $G(\bar{K}/K(x))$ fixes x, finite index

Cohomology of profinite groups

$G = \varprojlim G/H_i$ $\quad H_i = \ker(G \twoheadrightarrow G_i)$, hence $G_i \cong G/H_i$ (surjective)

where H_i finite index

$M = \varinjlim M^{H_i}$

want to make $H^q(G/H_i, M^{H_i})$ into directed system.

Let $i \leq j$ – there is a map $G/H_j \to G/H_i$, so $H_j \subset H_i$

$$H^q(G/H_i, M^{H_i}) \to H^q(G/H_j, M^{H_j})$$

$$[c_i] \longmapsto [c_j]$$

$$(G/H_j)^q \to (G/H_i)^q \xrightarrow{\text{cocycle } c_i} M^{H_i} \hookrightarrow M^{H_j}$$

$$(G/H_j)^q \xrightarrow{c_j} M^{H_j}$$

need to check works up to coboundary

Def. $H^q(G, M) = \varinjlim H^q(G/H_i, M^{H_i})$

Not necessarily the same as the derived functor cohomology

Alternative: Look at continuous maps $G^q \to M$ (discrete topology)

take $\frac{\text{continuous cocycles}}{\text{continuous coboundaries}}$, also gives $H^q(G, M)$

Example. $H^q(G(\bar{K}/K), \bar{K}^+) = \begin{cases} K & q=0 \\ 0 & q>0 \end{cases}$

$H^q(G(\bar{K}/K), \bar{K}^*) = 0$

follows from results for finite groups – limit of trivial groups is trivial

Def, The Brauergroup of K is $Br(K) = H^2(G(\bar{K}/K), \bar{K}^*)$

Theorem $K/\mathbb{Q}_p$ a finite extension, then

$$Br(K) \cong \mathbb{Q}/\mathbb{Z}$$

Pf Sketch, K_{nr} = Max unramified extension of K

① $H^2(G(\overline{K}/K_{nr}), \overline{K}^\times) = 0$ (won't prove)

② use inflation / restriction

$$\left[\begin{array}{l} H^q(L/K, M) \text{ means } H^q(Gal(L/K), M) \\ H^q(K, M) \text{ means } H^q(\overline{K}/K, M) \\ \quad \shortparallel \\ H^q_{et}(Spec(K), M) \end{array}\right]$$

$$0 \to H^1(K_{nr}/K, K_{nr}^\times) \xrightarrow{inf} H^1(\overline{K}/K, \overline{K}^\times) \xrightarrow{res} H^1(\overline{K}/K_{nr}, \overline{K}^\times) -$$

but Hilbert 90 ⟹ these are all 0. Luckily this implies

$$0 \to H^2(K_{nr}/K, K_{nr}^\times) \xrightarrow{inf} H^2(\overline{K}/K, \overline{K}^\times) \xrightarrow{res} H^2(\overline{K}/K_{nr}, \overline{K}^\times)$$

Step ① ↑ this is 0

thus $Br(K) = H^2(K_{nr}/K, K_{nr}^\times)$

③ $1 \to \mathcal{O}_{nr}^\times \to K_{nr}^\times \xrightarrow{valuation} \mathbb{Z} \to 0$

valuation on $\overline{K}^\times$ has to go to $\mathbb{Q}$

this induces $H^2(\mathcal{O}_{nr}^\times) \to H^2(K_{nr}^\times) \to H^2(\mathbb{Z}) \to H^3(\mathcal{O}_{nr}^\times)$

(4) $H^q(\mathcal{O}_{nr}^\times) = 0 \quad \forall q \geq 1$ (we'll do next time)

(5) Step 4 $\Rightarrow H^2(K_{nr}^\times) \cong H^2(\mathbb{Z})$

Consider $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

induces $\to H^1(\mathbb{Q}) \to H^1(\mathbb{Q}/\mathbb{Z}) \to H^2(\mathbb{Z}) \to H^2(\mathbb{Q})$

Lemma G profinite, $H^q(G, \mathbb{Q}) = 0 \quad \forall q \geq 1$

(6) $H^2(\mathbb{Z}) \cong H^1(G(K_{nr}/K), \mathbb{Q}/\mathbb{Z}) \cong$
(trivial action $\Rightarrow$ 1-cycles just homomorphisms)

$\mathrm{Hom}_{\mathrm{cont}}(G(K_{nr}/K), \mathbb{Q}/\mathbb{Z})$

$f \longmapsto f(\mathrm{Frob}_u)$

$\| \wr$

$G(\bar{u}/u)$ where u = residue field of K

$\| \wr$

$\hat{\mathbb{Z}}$ topologically cyclic

here $\cong \mathbb{Q}/\mathbb{Z}$

Lecture 18 (2011-03-11)

$K/\mathbb{Q}_p$, want to compute $Br(K) \overset{\text{def}}{=} H^2(\bar{K}/K, \bar{K}^\times)$ 2011-3-11 (18)

we showed it sufficed to look at $H^2(K_{nr}/K, K_{nr}^\times)$

we showed this is isomorphic to $H^2(K_{nr}/K, \mathbb{Z})$ via the map induced by the valuation map on K_{nr}

this is isomorphic to $H^1(K_{nr}/K, \mathbb{Q}/\mathbb{Z})$ by dimension shifting

this is isomorphic to $H^1(\bar{k}/k, \mathbb{Q}/\mathbb{Z}) \cong H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$

$\xrightarrow{\text{evaluation}} \mathbb{Q}/\mathbb{Z}$

Prop. $H^i(K_{nr}/K, \mathcal{O}_{nr}^\times) = 0$ for $i \geq 1$

If. $1 \to 1 + \mathfrak{p}_{nr} \to \mathcal{O}_{nr}^\times \to \bar{k}^\times \to 1$

Let L/K be a finite unramified extension.

To prove: $H^i(L/K, \mathcal{O}_L^\times) = 0$ (get full result by taking direct limits)

$1 \to 1 + \mathfrak{p}_L \to \mathcal{O}_L^\times \to k_L^\times \to 1$

$G_{L/K} \cong G_{k_L/k_K}$ is cyclic

$H^1(k_L/k_K, k_L^\times) = 0$ by Hilbert 90

$U_L^\times$ is finite, so $H^1($

Herbrand quotient $h(U_L^\times) = 1$

$$1 \to 1 + \mathfrak{p}_K \to O_K^\times \to U_K \to H^1(1+\mathfrak{p}_L) \to H^1(O_L^\times) \to$$

we know this is onto, so this is all 0

This shows $H^i(L/K, 1+\mathfrak{p}_L) \simeq H^i(L/K, O_L^\times)$

Prop. $H^2(K_{nr}/K, O_{nr}^\times) = 0$ Let $\mathfrak{p}_L = \pi_L O_L$

$$1 \to 1 + \mathfrak{p}_L^{n+1} \to 1 + \mathfrak{p}_L^n \to U_L^+ \to 0$$

$$1 + a\pi_L^n \mapsto a \bmod \mathfrak{p}_L$$

$$(1 + a\pi_L^n)(1 + b\pi_L^n) = 1 + (a+b)\pi_L^n + \cdots$$

$H^i(K_t/K_{ur}, U_L^+) = 0$ from before

So $H^i(L/K, 1+\mathfrak{p}_L^n)$ is independent of n.

When n is big enough, we have an isomorphism

$$(1 + \mathfrak{p}_L^n, \cdot) \xrightarrow{\sim} (\mathfrak{p}^n, +)$$ where $p \geq 3$, any $n \geq 1$; $p = 2$, $n \geq 2$

$$\exp(a) = \sum_{k=0}^{\infty} \frac{a^k}{n!} \longleftarrow\!\shortmid\; a$$

$$b \longmapsto \log(b) = -\sum_{k=0}^{\infty} \frac{(b-1)^k}{k}$$

Thus $H^i(L/K, 1+\mathfrak{P}_L^{\wedge}) = H^i(L/K, \mathfrak{P}_L^{\wedge})$

Because $\mathfrak{P}_L^{\wedge} \cong \mathcal{O}_L$, we get

$a \mapsto \frac{a}{\pi_L^{\wedge}}$ (not canonical)

$\cong H^i(L/K, \mathcal{O}_L^+)$

Since these are local fields,

$$\mathcal{O}_L = \mathcal{O}_K[x]/(f(x))$$

where $f(x)$ is monic, irreducible over K

$$f(x) = \prod (x - a_i) = \prod_{\sigma \in G_{L/K}} (x - \sigma(\alpha))$$

$$\mathcal{O}_L^+ = \bigoplus_{\sigma \in G_{L/K}} \mathcal{O}_K \cdot \sigma(\alpha) \cong \mathbb{Z}[G_{L/K}] \otimes_{\mathbb{Z}} \mathcal{O}_K$$

$$\sigma(\alpha) a \longleftrightarrow \sigma \otimes a$$

$\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G_{L/K}], \mathcal{O}_K)$ since $G_{L/K}$ finite

co-induced by definition, so $H^i(\ ) = 0$ $\forall i \geq 1$

Thus $H^i(L/K, \mathcal{O}_L^{\times}) = 0 \quad \forall i \geq 1$

$H(\mathcal{O}_L^\times) \cong H(1+\mathfrak{p}_L) \cong H(1+\mathfrak{p}_L^n) \cong H(\mathfrak{p}_L^n) \cong H(\mathcal{O}_L) = 0$ (exp, log)

$Br(K) \overset{\text{def}}{=} H^2(\overline{K}/K, \overline{K}^\times) \cong H^2(K_{nr}/K, K_{nr}^\times) \xrightarrow{\sim} H^2(K_{nr}/K, \mathbb{Z}) \xleftarrow[\sim]{\delta} H^1(K_{nr}/K, \mathbb{Q}/\mathbb{Z}) \cong H^1(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$

called "invariant map" — evaluation

$Br(K) \xrightarrow[\sim]{inv_K} \mathbb{Q}/\mathbb{Z}$ functorial

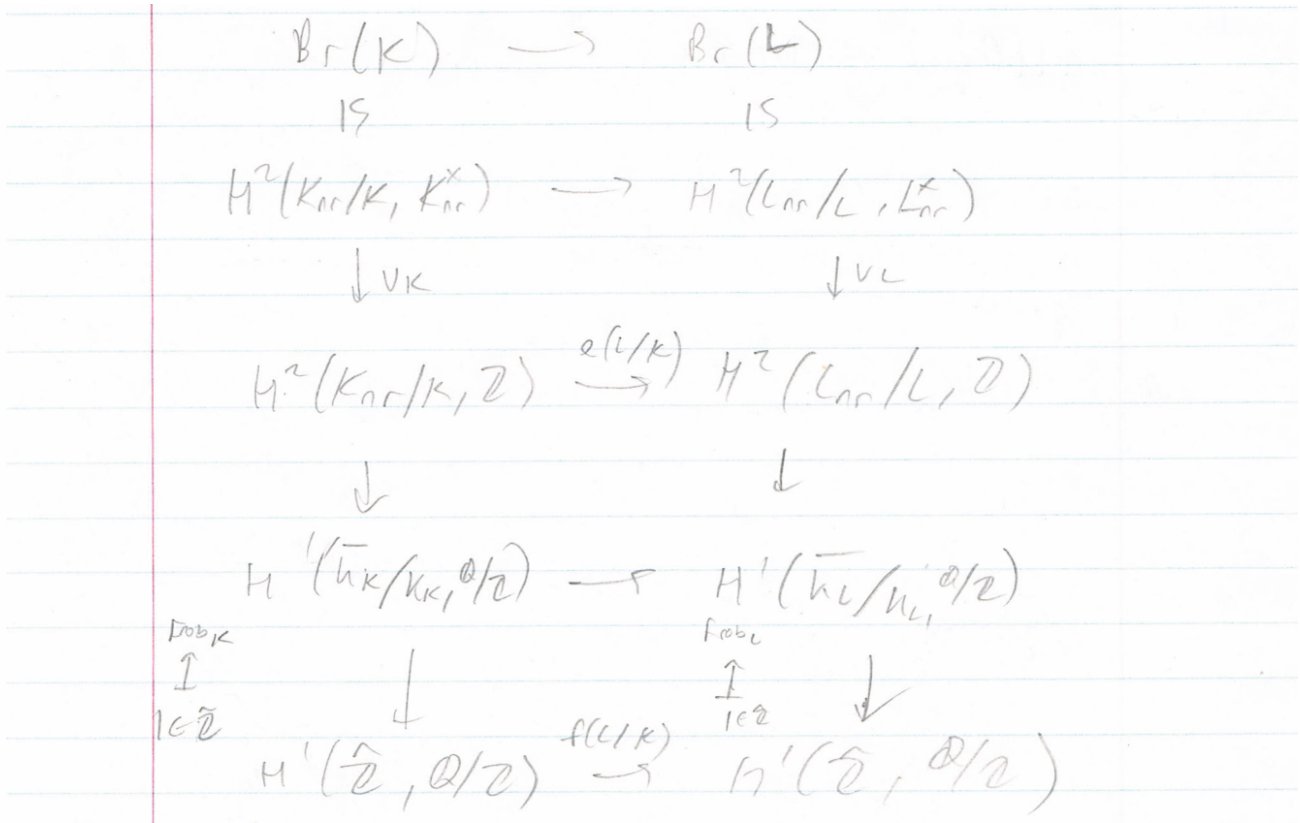
$L/K/\mathbb{Q}_p$ finite, $G(\overline{L}/L) \subset G(\overline{K}/K)$

all H^1's are trivial by Hilbert 90, so

get inflation/restriction on H^2's

$$0 \to H^2(L/K, L^\times) \xrightarrow{inf} H^2(\overline{K}/K, \overline{K}^\times) \xrightarrow{res} H^2(\overline{L}/L, \overline{L}^\times)$$

$$\begin{array}{ccccc} & & inv_{L/K} \downarrow & inv_K \downarrow s & inv_L \downarrow s \\ 0 \to & & ? & \mathbb{Q}/\mathbb{Z} & \longrightarrow \mathbb{Q}/\mathbb{Z} \\ & & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & & \end{array}$$



so this map is multiplication by $e(L/K)f(L/K) = [L:K]$ all thru

Def. The invariant of L/K is

$$u_{L/K} = \mathrm{inv}_{L/K}^{-1}\left(\frac{1}{n}\right) \in H^2(L/K, L^\times)$$

(distinguished generator)

Lecture 19 (2011-03-14)

Group Homology 2011-3-14

(19)

cohomology = derived functor of $M \mapsto M^G$

M^G = largest submodule fixed by G

homology = derived functor of $M \mapsto M_G$

M_G = largest quotient fixed by G

Recall $M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.

Let I_G = left ideal generated by all $\sigma - 1$, for $\sigma \in G$

Then $M_G = M/I_G M = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/I_G$

$H_q(G, \cdot)$ = q^{th} derived functor of $\cdot \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/I_G$

Prop. $H_1(G, \mathbb{Z}) \cong I_G/I_G^2$

Pf. $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$

$\sigma \mapsto 1$

$$\to H_1(G, \mathbb{Z}[G]) \to H_1(G, \mathbb{Z}) \to H_0(G, I_G) \to H_0(G, \mathbb{Z}[G]) \to H_0(G, \mathbb{Z}) \to 0$$

$$\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$$

$$0 \qquad I_G/I_G^2 \xrightarrow{0 \text{ map}} \mathbb{Z}[G]/I_G \qquad \mathbb{Z}/I_G\mathbb{Z}$$

$$\| \qquad\qquad\qquad\qquad \| \qquad\qquad \|$$

$$\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$$

Prop $H_1(G,\mathbb{Z}) \cong I_G/I_G^2 \cong G^{ab} = G/[G,G]$

$\sigma - 1 \longleftarrow \sigma \bmod [G,G]$ (HW)

Tate cohomology groups G finite

Let $N = \sum_{\sigma \in G} \sigma \in \mathbb{Z}[G]$

Def $\hat{H}^q(G,M) = H^q(G,M)$ for $q \geq 1$

$\hat{H}^0(G,M) = M^G/NM$

$\hat{H}^{-1}(G,M) = \ker(M_G \xrightarrow{N} M_G)$

$\hat{H}^{-q}(G,M) = H_q(G,M)$ for $q \geq 2$

Thm. $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$ gives a long exact sequence

$$\to \hat{H}^q(G,M) \to \hat{H}^q(G,N) \to \hat{H}^q(G,P) \to \hat{H}^{q+1}(G,M) \to \cdots$$

need to check

$$\begin{array}{ccc} \hat{H}^{-1}(G,P) & \to & \hat{H}^0(G,M) \\ \| & & \| \\ \ker(P_G \xrightarrow{N} P_G) & \longrightarrow & M^G/N\cdot M \end{array}$$

$\bar{p} \in P/I_G P$

Choose $n \in N$ with $\beta(n) = p$ (since β surjective)

We know $N \cdot p \in I_G P$

$N \cdot p(n) \in I_G P$ $\quad p(N \cdot n) \in I_G p$ $\quad Np - \sum a_\sigma (\sigma - 1) p_\sigma$ ⋮ n_σ

...

Cup product $\hat{H}^p(G, M) \otimes \hat{H}^q(G, N) \longrightarrow \hat{H}^{p+q}(G, M \otimes N)$

$a \otimes \beta \longmapsto \alpha \cup \beta$

Functorial in M and N, respects connecting homomorphisms $\delta \alpha \cup \beta = \delta(\alpha \cup \beta)$ etc, and for $p = q = 0$ induced by $M^G \otimes N^G \longrightarrow (M \otimes N)^G$

For $p, q \geq 1$, we have

$$\bar{\alpha} \in \hat{H}^p(G, M), \quad \bar{\beta} \in \hat{H}^q(G, N)$$

Choose cocycles $\alpha : \mathbb{Z}[G^{p+1}] \to M$, $\beta : \mathbb{Z}[G^{q+1}] \to N$

Then $\alpha \cup \beta$ is the cohomology class for

$$\mathbb{Z}[G^{p+q+1}] \to M \otimes N$$

$$(\alpha \cup \beta)(\sigma_0, \ldots, \sigma_{p+q}) = \alpha(\sigma_0, \ldots, \sigma_p) \otimes \beta(\sigma_{p+1}, \ldots, \sigma_{p+q})$$

Properties:

- $(\bar{\alpha} \cup \bar{\beta}) \cup \bar{\gamma} = \bar{\alpha} \cup (\bar{\beta} \cup \bar{\gamma})$
- $\bar{\alpha} \cup \bar{\beta} = (-1)^{\dim(\bar{\alpha}) \dim(\bar{\beta})} \bar{\beta} \cup \bar{\alpha}$
- $\mathrm{Res}(\bar{\alpha} \cup \bar{\beta}) = \mathrm{Res}(\bar{\alpha}) \cup \mathrm{Res}(\bar{\beta})$ for Res = restriction (H ⊂ G)

Recall that for $L/K/\mathbb{Q}_p$ we defined

$$u_{L/K} \in H^2(L/K, L^\times) \cong \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

Thm. For all q,

$$\hat{H}^q(G_{L/K}, \mathbb{Z}) \xrightarrow{\cdot \smile u_{L/K}} \hat{H}^{q+2}(G_{L/K}, L^\times)$$

is an isomorphism.

Take $q = -2$

$$\begin{array}{ccc} \hat{H}^{-2}(G_{L/K}, \mathbb{Z}) & \xrightarrow{\smile u_{L/K}} & \hat{H}^0(G_{L/K}, L^\times) \\ \| & & \| \\ H_1(G_{L/K}, \mathbb{Z}) & & K^\times / N_{L/K} L^\times \\ \| & \nearrow \sim & \\ G_{L/K}^{ab} & & \end{array}$$

inverse to the Artin map

Lecture 20 (2011-03-16)

$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$ 2011-3-16 (20)

$N = \sum_{\sigma \in G} \sigma$ $\quad I_G = (\sigma - 1)_{\sigma \in G}$

$\hat{H}^{-1}(P) \to \hat{H}^0(M)$

$\| \text{def}$ $\quad \| \text{def}$

$\ker(H_0(P) \xrightarrow{N} H^0(P))$ $\quad M^G / NM$

$\ker(P/I_G P \xrightarrow{N} P^G)$ Let $\bar{p} \in P/I_G P$ such that $Np = 0$.

Choose $n \in \beta^{-1}(p)$. Then $\beta(Nn) = N\beta(n) = Np = 0$,

so $\exists!\, m \in M$ with $\alpha(m) = Nn$, and set $\delta(\bar{p}) = \bar{m}$

C_K = ideal class group $= I_K / K^\times$

L/K finite Galois $\quad I_K = \prod'_{w \in M_K} K_w^\times$

Thm. $H^2(G_{L/K}, C_L) \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ where $n = [L:K]$

$U_{L/K} = \text{inv}_{L/K}^{-1}\left(\frac{1}{n}\right)$ $\quad K_v^\times \cong \left(\prod_{w|v} L_w^\times\right)$

$1 \to L^\times \to I_L \to C_L \to 1$

$1 \to K^\times \to I_K \to C_L^{G_{L/K}} \to H^1(G_{L/K}, L^\times)$

$\|$ Hilbert 90

0

Thm L/K finite Galois number fields

$$\hat{H}^q(G_{L/K}, \mathbb{Z}) \xrightarrow{\cup u_{L/K}} \hat{H}^{q+2}(G_{L/K}, C_L)$$

is an isomorphism

Taking $q = -2$:

$$G_{L/K}^{ab} \xrightarrow{\cup u_{L/K}} C_K / N_{L/K} C_L$$

Take limit over all abelian extensions

$$G_K^{ab/K} \cong \varprojlim C_K / N_{L/K} C_L = C_K / \{\text{universal norms}\}$$

$$\{\alpha \in C_K \mid \forall\, L/K \text{ abelian}, \exists\, \beta_L \in C_L \text{ with } N_{L/K}\beta_L = \alpha_L\}$$

$$\begin{array}{ccccccc} 0 \to & H^2(G_{L/K}, C_K^\times) & \to & H^2(G_{\bar{K}/K}, C_{\bar{K}}) & \to & H^2(G_{\bar{L}/L}, C_{\bar{L}}) \\ & \downarrow \mathrm{inv}_{L/K} & & \downarrow \mathrm{inv}_K & & \downarrow \mathrm{inv}_L \\ 0 \to & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \to & \mathbb{Q}/\mathbb{Z} & \to & \mathbb{Q}/\mathbb{Q} \end{array}$$

HW: $Br(\mathbb{C}) = 0$, $Br(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$

$Br(K) = H^2(G_{\bar{K}/K}, \bar{K}^\times)$

$1 \to \bar{K}^\times \to I_K \to C_K \to 1$

$H^1(I_K) \to H^1(C_K) \to H^2(\bar{K}^\times) \to H^2(I_K) \to H^2(C_K) \to H^3(\bar{K}^\times)$

$0 \quad 0 \quad Br(K) \quad \bigoplus_{v \in M_K} (\mathbb{Q}/\mathbb{Z})_v \quad \mathbb{Q}/\mathbb{Z}$

Hilbert 90

$= 0$ if v real
$= \frac{1}{2}$ if v complex

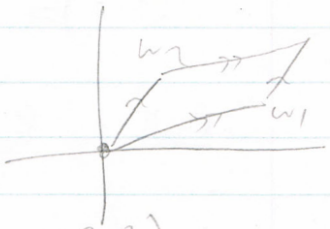
Elliptic curves

Goals: (weak) Mordell-Weil Thm
Complex multiplication

Def: An elliptic curve E is a smooth, projective curve of genus 1, with given point O.

Look at as Riemann surface

$\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = S^1 \times S^1$



Using Riemann-Roch can be embedded in $\mathbb{P}^2$ as a cubic

$E = \{Y^2Z = X^3 + AXZ^2 + BZ^3\}$

for some A, B (not true if $char = 2, 3$)

Here $O = [0, 1, 0]$ is unique point "at infinity"

We homogenize, get $E: y^2 = x^3 + Ax + B$

Def. This is called a Weierstrass Equation for E

Prop The curve $y^2 = x^3 + Ax + B$ is smooth if and only if $\Delta_E \neq 0$

Def: The discriminant of E is $\Delta_E = -16(4A^3 + 27B^2)$

Def. E is defined over K if $A, B \in K$

We write E/K if E defined over K

$$E(L) = \{(x,y) \in L^2 \mid y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$$

Example $E: y^2 = x^3 + 1$

$$E(\mathbb{Q}) = \{\mathcal{O}, (0, \pm 1), (-1, 0), (2, \pm 3)\}$$

↑ not so easy to prove

$$\bar{K}(E) = \mathrm{Frac}\left(\bar{K}[x,y] / (y^2 - x^3 - Ax - B)\right)$$

Def. $E \cong E'$ if $\exists$ maps

$\phi: E \to E'$ $\quad \phi \circ \psi = id$, $\psi \circ \phi = id$

$\psi: E' \to E$

ϕ, ψ are rational functions in $\bar{K}(x,y)$

$E \cong E'$ over K if can choose ϕ, ψ with coeffs in K

Lecture 21 (2011-03-18)

$E: y^2 = x^3 + Ax + B$ 2011-3-18

(21)

$E \cong E' \iff E': y^2 = x^3 + Au^4x + Bu^6$

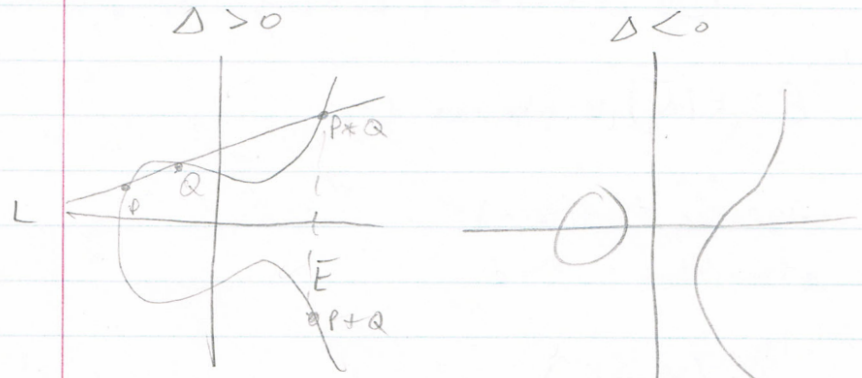
$\frac{A^3}{B^2}$ is an isomorphism invariant

Def. The j-invariant of E is $j(E) = 1728 \frac{4A^3}{-16(4A^3+27B^2)}$

Prop. $E \cong E'$ iff $j(E) = j(E')$, when working over an algebraically closed field.

Prop. $\forall \alpha \in K$, $\exists E/K$ with $j(E) = \alpha$ (K alg closed)

Elliptic curves have a group structure (group object in category of algebraic varieties)



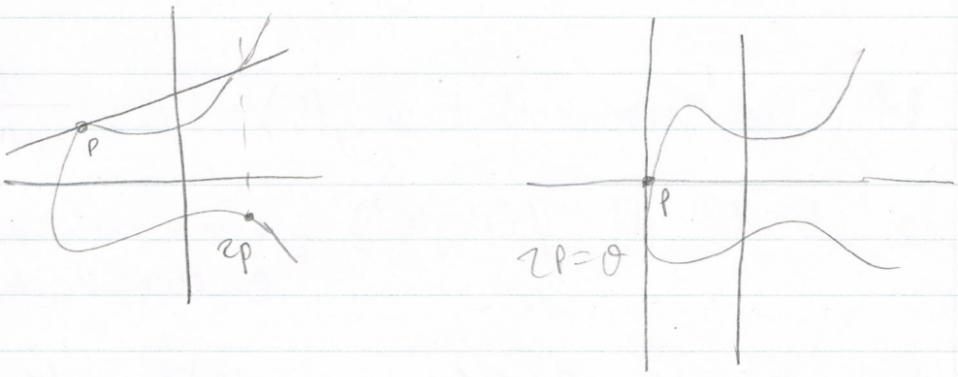
$E \cap L = 3$ pts. (Bezout)

$Q * (P * Q) = P$ for all P, Q

$-P$ = reflection about x-axis

Group law defined by

$$P + Q + P * Q = \mathcal{O}$$



$\frac{dy}{dx} = \frac{3x^2+A}{2y} = \infty \iff y = 0$, x not root of $3x^2+A$

Prop $\{P \in E(\mathbb{C}) \mid 2P = \mathcal{O}\} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Thm. $E = E(\bar{K})$ is abelian group

Pf abelian – trivial
associative – hard

$$-(x_0, y_0) = (x_0, -y_0)$$

If $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$, we have

(if $x_1 = x_2$, $y_1 = -y_2$ then $(x_3, y_3) = \mathcal{O}$)

Otherwise, line through (x_1, y_1) and (x_2, y_2)

$L: y = \lambda x + \nu$

$\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ $\qquad \lambda = \frac{3x_1^2 + A}{2y_1}$

$\nu = y_1 - \lambda x_1$ $\qquad \nu = y_1 - \lambda x_1$

$L \cap E: (\lambda x + \nu)^2 = x^3 + Ax + B$

$x^3 - \lambda^2 x^2 + (A - 2\lambda\nu)x + B - \nu^2$

$= (x - x_1)(x - x_2)(x - x_3)$

$-\lambda^2 = -x_1 - x_2 - x_3$ so $x_3 = \lambda^2 - x_1 - x_2$

$y_3 = -(\lambda x_3 + \nu)$

Corollary. Given E/K and field extension L/K,

$E(L) = \{(x, y) \in E \mid x, y \in L\} \cup \{\mathcal{O}\}$ is a subgroup of $E(\bar{K})$

Pf. Closed under $+$ by formula.

Example $E: y^2 = x^3 + 17$

$P = (-2, 3)$, $Q = (-1, 4)$, $R = (2, 5)$ are in $E(\mathbb{Q})$

$Q + R = \left(\frac{-8}{9}, -\frac{109}{27}\right)$ $\qquad 2P = (8, -23)$

HW: Compute $2Q$, $P+Q$

Thm. Let $E: y^2 = x^3 + 17$. Then

a) $E(\mathbb{Q}) = \{mP + nR \mid m, n \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}$

b) $|E(\mathbb{Z})| = 16$

Pf: Hard

These illustrate: Mordell-Weil Thm.

$K/\mathbb{Q}$ number field, E/K is an elliptic curve

$E(K)$ is finitely generated abelian group

$E(K) \cong (\text{finite}) \times \mathbb{Z}^r$ $r = \text{rank } E$

Goal: Weak Mordell-Weil Thm

$E(K)/mE(K)$ is finite.

Siegel's Thm

$E(\mathcal{O}_K)$ finite.

Conjecture: $\forall N \in \mathbb{N}$, $\exists E/\mathbb{Q}$ with $\mathrm{rank}\, E \geq N$

best so far: 28 ish, Noam Elkies

Def $E(K)_{tors} = \{P \in E(K) \mid \exists P = \mathcal{O} \text{ for some } n \geq 1\}$

Thm, Mazur: $|E(\mathbb{Q})_{tors}| \leq 16$ $\mathbb{Z}/n\mathbb{Z}$ for $1 \leq n \leq 10$

Merel: $|E(K)_{tors}| \leq C([K:\mathbb{Q}])$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for $1 \leq n \leq 4$

L-series

$\# E(\mathbb{F}_p) = p + 1 - a_p$

Lecture 22 (2011-03-21)

$E: y^2 = x^3 + Ax + B$ defined over K 2011-3-21

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$[m]: E \to E$

$P \mapsto \underbrace{P + \cdots + P}_{m \text{ times}}$ $\qquad E_{tors} = \bigcup_{m \geq 1} \ker[m]$

$E[m] = \ker[m] = \{P \in E(\bar{K}) \mid [m]P = \mathcal{O}\}$

Note that $[m]P = (f_m(x(P), y(P)), g_m(x(P), y(P)))$

where $f_m, g_m \in K(x, y)$

Ex. Work out f_2 explicitly. In fact, all f_m have only even powers of y, hence can be written in terms of only x.

$E[m]$ = points where denominators of f_m and g_m vanish

Notice that if $P \in E(\bar{K})$ and $\sigma \in G_{\bar{K}/K}$,

$\sigma P \overset{\text{def}}{=} \sigma(x, y) = (\sigma(x), \sigma(y)) \in E(\bar{K})$

$\sigma(P + Q) = \sigma(P) + \sigma(Q)$, so $E(\bar{K})$ is a $G_{\bar{K}/K}$-module

also $E[m]$ is a $G_{\bar{K}/K}$-module

Thm. E/K elliptic curve, $p = \mathrm{char}(K)$

a) If $p \nmid m$, or $p = 0$, then $E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

b) $E[p^n] \cong$ either $\mathbb{Z}/p^n\mathbb{Z}$ for all n, or 0 for all n (supersingular)

Prop. $\mathrm{Char}(K) = 0$ or $p \geq 3$,
$E: y^2 = x^3 + Ax + B$ (so not all curves in char 3)

then $E[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Pf $P \in E[2] \iff 2P = 0 \iff$ vertical tangent at $P \iff$

$\left.\frac{dx}{dy}\right|_P = 0 \iff \frac{2y}{3x^2+A} = 0 \iff y = 0$ (provided $3x^2 + A \neq 0$)

What happens if $y = 3x^2 + A = 0$ at $P = (x, y) \in E$?

On any curve $F(x, y) = 0$, a point (x, y) is singular if $F(x, y) = 0$, $\frac{\partial F}{\partial x}(x, y) = 0$, $\frac{\partial F}{\partial y}(x, y) = 0$

$F = y^2 - x^3 - Ax - B$; $\frac{\partial F}{\partial y} = -(3x^2 + A)$, $\frac{\partial F}{\partial y} = 2y$

but we assumed our curve was nonsingular, so

$E[2] = \{O\} \cup \{(x, y) \in E(\bar{K}) \mid y = 0\} =$
$\{O\} \cup \{(\alpha, 0) \mid \alpha^3 + A\alpha + B = 0\}$ has 4 elements
2-torsion $\Rightarrow$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$G_{\bar{K}/K}$ acts on $E[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

Get $\rho_{E,m} : G_{\bar{K}/K} \longrightarrow \mathrm{Aut}(E[m]) \cong GL_2(\mathbb{Z}/m\mathbb{Z})$,

a 2-dimensional representation of $G_{\bar{K}/K}$

Analogous to the 1-dimensional representation

$G_{\bar{K}/K} \longrightarrow \mathrm{Aut}\left(\underbrace{(\ker : \mathbb{C}^\times \to \mathbb{C}^\times)}_{x \mapsto x^m} \right)$

$\mu_m \cong \mathbb{Z}/m\mathbb{Z}$

$\sigma \longmapsto (\zeta \mapsto \sigma(\zeta))$

$K(E[m]) = K(x, y : (x,y) \in E[m])$

$G_{\bar{K}/K}$ maps $K(E[m])$ to itself, so $K(E[m])$ is Galois

$\rho_{E,m} : G(K(E[m])/K) \hookrightarrow GL_2(\mathbb{Z}/m\mathbb{Z})$

Weak Mordell-Weil Theorem

$K/\mathbb{Q}$, E/K, the $E(K)/mE(K)$ finite

to prove full MW theorem, suffices to prove weak for $m = 2$

Elliptic Kummer sequence $[m](x,y) = (f_m(x,y), g_m(x,y))$

$$0 \to E[m] \to E(\bar{K}) \xrightarrow{[m]} E(\bar{K}) \to 0$$

is an exact sequence of $G_{\bar{K}/K}$-modules, take cohomology

$$0 \to E[m] \cap E(K) \to E(K) \xrightarrow{[m]} E(K) \to H^1(G_{\bar{K}/K}, E[m]) \to H^1(G_{\bar{K}/K}, E$$

get $0 \to E(K)/mE(K) \to H^1(G_{\bar{K}/K}, E[m]) \to H^1(G_{\bar{K}/K}, E$

life is easier if $E[m] \subset E(K)$ since then $G_{\bar{K}/K}$ action on $E[m]$ is trivial and this turns into Hom

reduce to this case. Let $L = K(E[m])$.

Suppose $E(L)/mE(L)$ finite. To show: $E(K)/mE(K)$ finite.

$$0 \to \frac{E(K) \cap mE(L)}{mE(K)} \to E(K)/mE(K) \to E(L)/mE(L)$$

$$\downarrow \gamma \qquad \downarrow \qquad \downarrow$$

$$0 \to H^1(G_{L/K}, E[m]) \to H^1(G_{E/K}, E[m]) \to H^1(G_{\bar{L}/L}, E[m])$$

claim: bottom is actually part of the inflation-restriction sequence

$$0 \to H^1(G_{L/K}, E[m]) \xrightarrow{\text{inf}} H^1(G_{\bar{K}/K}, E[m]) \xrightarrow{\text{res}} H^1(G_{\bar{L}/L}, E[m]$$

Def of ψ. Let $P \in \bar{E}(K) \cap mE(L)$.

So $P = mQ$ for some $Q \in E(L)$

$$\psi(P)(\sigma) = \sigma(Q) - Q$$

If $E(L)/mE(L)$ finite, then because

$H^1(G_{L/K}, E[m])$ finite (because $G_{L/K}$ and $E(m)$ finite)

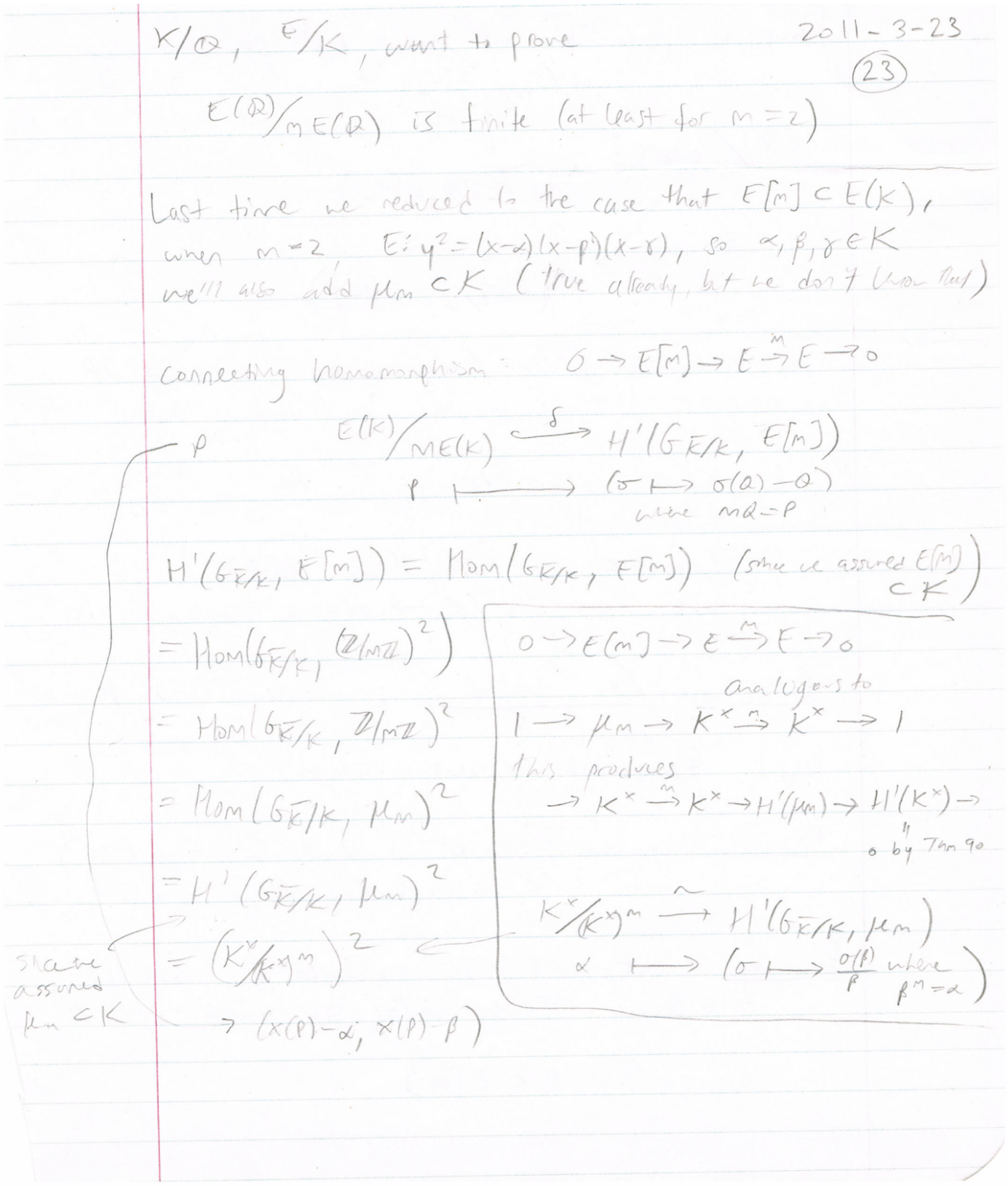
and ψ injective, $\frac{E(K) \cap mE(L)}{mE(K)}$ finite, hence

$E(K)/mE(K)$ finite [in general $H^1(G_{\bar{K}/K}, E[m])$ not finite

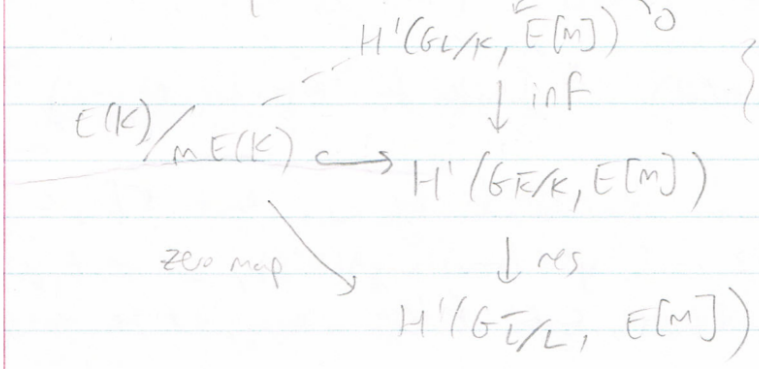
$= H^1(G_{\bar{K}/K}, E)$

$E)[m] \to 0$

Lecture 23 (2011-03-23)



want to replace $\bar{K}$ with smallest field possible, L



the cocycle $(\sigma \mapsto (\sigma(Q)-Q))$ is trivial iff σ acts trivially on Q, so want to define

$L = K(Q : mQ \in E(K))$ the

$E(K)/mE(K) \hookrightarrow H^1(G_{L/K}, E[m])$

$P \longmapsto (\sigma \mapsto \sigma(Q)-Q, \text{ where } mQ = P)$

Claim. L/K finite ; follows from Hurwitz's theorem, and

Prop 1. $G_{L/K}$ has exponent dividing m

Prop 2. There is a finite set of primes $S \subset \mathrm{Spec}(\mathcal{O}_K)$ such that L/K is unramified for all $\mathfrak{p} \notin S$.

Pf of Prop 1: Let $Q \in E(L)$ with $mQ = P$, and let $\sigma \in G_{L/K}$. Then $m(\sigma(Q)) = \sigma(mQ) = \sigma(P) = P$

thus $\sigma(Q) - Q \in E[m]$

Now look at $\sigma^m(Q) - Q$. Can write as

$$(\sigma^{m-1} + \cdots + 1)(\sigma - 1)Q = m(\sigma - 1)(Q) = O$$

$\sigma(Q) \in E[m]$, $E[m] \subset E(K)$

Prop 2. We defined $L = K([m]^{-1}E(K))$.

Fix $Q \in [m]^{-1}E(K)$, and let $F = K(Q)$.

$E: y^2 = x^3 + Ax + B$, $\mathfrak{p}$ prime in F,

reduce mod $\mathfrak{p}$ (assume A, B are $\mathfrak{p}$-integral)

$\tilde{A}, \tilde{B} \in \mathbb{F}_{\mathfrak{p}}$, $\tilde{E}: y^2 = x^3 + \tilde{A}x + \tilde{B}$

$\tilde{E}$ is an elliptic curve $/\mathbb{F}_{\mathfrak{p}} \iff \mathfrak{p} \nmid \Delta = 16(4A^3 + 27B^2)$
$\iff \tilde{\Delta} = 0$

Def. E has good reduction at $\mathfrak{p}$ if $\mathrm{ord}_{\mathfrak{p}}(A), \mathrm{ord}_{\mathfrak{p}}(B) \geq 0$ and $\mathfrak{p} \nmid \Delta$

Assume good reduction; then $\tilde{E}(\mathbb{F}_{\mathfrak{p}})$ is a group, and get a reduction map

$$E(F) \longrightarrow \tilde{E}(\mathbb{F}_{\mathfrak{p}})$$
$$(x, y) \mapsto \begin{cases} (\tilde{x}, \tilde{y}) & \text{if } \mathrm{ord}_{\mathfrak{p}}(x) \geq 0 \ (\text{which} \Rightarrow \mathrm{ord}_{\mathfrak{p}}(y) \geq 0) \\ O & \text{else} \end{cases}$$

This is just $\mathbb{P}^n(F) \to \mathbb{P}^n(\mathbb{F}_p)$

$[x_0, \dots, x_n] \mapsto [\widetilde{\lambda x_0}, \dots, \widetilde{\lambda x_n}]$

where we choose $\lambda \in F^\times$ such that $\operatorname{ord}_p(\lambda x_i) \geq 0$ for all i

and $\operatorname{ord}_p(\lambda x_j) = 0$ for some j.

Prop. $E(F) \to \widetilde{E}(\mathbb{F}_p)$ is a group homomorphism

Pf. The group law is defined by $P+Q+R=\mathcal{O} \iff$ P, Q, R collinear, and reduction mod p takes lines to lines

QED (at least if $\widetilde{P}, \widetilde{Q}, \widetilde{R}$ distinct)

Lecture 24 (2011-03-25)

Goal: $K([m]^{-1}E(K))/K$ is finite. 2011-3-25

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If $\mathfrak{p} \nmid \Delta$, get a homomorphism $E(K) \to \tilde{E}(\mathbb{F}_{\mathfrak{p}})$.

Key Lemma. If $\mathfrak{p} \nmid m\Delta$, then $E(K)[m] \hookrightarrow \tilde{E}(\mathbb{F}_{\mathfrak{p}})$

(Analogous to statement that roots of unity remain distinct mod $\mathfrak{p}$)

We only need this for $m = 2$ though. Take Weierstrass equation with coefficients in $\mathcal{O}_K$, then factor as

$E: y^2 = (x-\alpha)(x-\beta)(x-\gamma)$. Then $E[2] = \{\mathcal{O}, (\alpha,0), (\beta,0), (\gamma,0)\}$

maps to $\tilde{E}(\mathbb{F}_{\mathfrak{p}})$ then $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are distinct mod $\mathfrak{p}$.

But $\Delta = \pm 16(\alpha-\beta)^2(\beta-\gamma)^2(\alpha-\gamma)^2$, so $\mathfrak{p} \nmid \Delta$ gives the result.

Prop. Assume $\mathfrak{p} \nmid m\Delta$. Then

a) $K(E[m])/K$ is unramified at $\mathfrak{p}$.

b) Let $Q \in E(\bar{K})$ with $mQ \in E(K)$, then $K(Q)/K$ is unramified at $\mathfrak{p}$.

Pf. Let $\mathfrak{P}$ be a prime of $K(E[m])$, $\mathfrak{P}/\mathfrak{p}$. Then

$$I(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G_{\mathfrak{P}} \mid \sigma(\alpha) = \alpha \bmod \mathfrak{P}\ \forall \alpha\} = \{\sigma \in G_{\mathfrak{p}} \mid \sigma(T) = T \bmod \mathfrak{P}\ \forall T \in E[m]\}$$

$$= \{\sigma \in G_{\mathfrak{p}} \mid \forall T \in E[m],\ \sigma(T) - T \in \ker(E[m] \to \tilde{E} \bmod \mathfrak{p})\} = \{\sigma \in G_{\mathfrak{p}} \mid \sigma(T) = T\ \forall T \in E[m]\} = \{1\}$$

(Lemma says injective)

b) WLOG, using a, we can replace K by $K(E[m])$ (unramified + unramified = unramified).

Let $\mathfrak{P}$ be a prime of $K(\bar{Q})$, $\mathfrak{P} \mid \mathfrak{p}$. Then

$$I(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G_{\mathfrak{p}} \mid \sigma(Q) \equiv Q \bmod \mathfrak{p}\} = \{\sigma \in G_{\mathfrak{p}} \mid \sigma(Q) - Q \equiv 0 \bmod \mathfrak{p}\}$$

Note $m(\sigma(Q) - Q) = \sigma(mQ) - mQ = 0$ (because $mQ \in E(K)$, hence fixed by σ)

So $\sigma(Q) - Q \in E[m]$, so $I(\mathfrak{P}/\mathfrak{p}) = \{\sigma \in G_{\mathfrak{p}} \mid \sigma(Q) - Q = 0\} = \{1\}$.

Corollary. Given E/K, assume $E[m] \subset E(K)$. Let

$S = \{\text{primes } \mathfrak{p} \mid m\Delta\}$. Then.

a) Then $K([m]^{-1}E(K))/K$ is unramified at every $\mathfrak{p} \notin S$ (we showed this for any one point in this, composita of unramified are unramified)

b) $\mathrm{Gal}(K([m]^{-1}E(K))/K)$ is abelian of exponent m.

Pf. a) from prop.

b). Exponent m last time. Pf of abelian?

Let $Q \in E(\bar{K})$, $mQ = P \in E(K)$. Let $\sigma, \tau \in G(K(Q)/K)$

$\sigma(Q) - Q = T_\sigma$ for some $T_\sigma \in E[m]$

$\tau(Q) - Q = T_\tau$ for some $T_\tau \in E[m]$

$\sigma(\gamma(Q)) = \sigma(Q + T_\gamma) = Q + T_\sigma + \underbrace{\sigma(T_\gamma)}_{= T_\gamma \text{ since } E(M) \subset E(K)}$

similarly $\gamma(\sigma(Q)) = Q + T_\gamma + T_\sigma$

Thm. $K/\mathbb{Q}$, S = finite set of primes, $m \geq 2$

Let L = max abelian extension of K of exponent m unramified outside S. Then L/K is finite.

Main Thm of Kummer Theory $K/\mathbb{Q}$, $m \geq 2$, $\mu_m \subset K$

M = max abelian extension of K of exponent m

Then $M = K(\sqrt[m]{\alpha} \mid \alpha \in K^\times)$.

Pf of Thm. WLOG, can expand S, so $S \supset \{p \mid m\}$.

WLOG, can assume $\mu_m \subset K$.

L = largest subfield of M unramified outside S

When is $K(\sqrt[m]{\alpha})$ ramified at p? (assume $p \nmid m$)

$x^m - \alpha = 0$, $\operatorname{disc}(x^m - \alpha) = \pm m^? \cdot \alpha^{m-1}$

Let π be a uniformizer at p (i.e. $\operatorname{ord}_p(\pi) = 1$), so

$\alpha = u \pi^r$, $\sqrt[m]{\alpha} = \sqrt[m]{u}\, \pi^{r/m}$, so $K(\sqrt[m]{\alpha})/K$ unramified at p
(u unit) $\iff m \mid \operatorname{ord}_p(\alpha)$

Let $T_S = \{\alpha \in K^\times/(K^\times)^m \mid \mathrm{ord}_\mathfrak{p}(\alpha) \equiv 0 \bmod m \text{ for all } \mathfrak{p} \in S\}$

Then $L = K(\sqrt[m]{\alpha} : \alpha \in T_S)$.

Goal: show T_S is a finite set.

$$0 \to T_\phi \to T_S \longrightarrow \underbrace{(\mathbb{Z}/m\mathbb{Z} \times \cdots \times \mathbb{Z}/m\mathbb{Z})}_{\#S} \quad \text{finite}$$
$$\alpha \longmapsto (\mathrm{ord}_\mathfrak{p}(\alpha))_{\mathfrak{p} \in S}$$

Goal: T_ϕ is a finite set.

$\alpha \in T_\phi \iff \mathrm{ord}_\mathfrak{p}(\alpha) \equiv 0 \bmod m \ \forall \mathfrak{p} \iff$

$\alpha \mathcal{O}_K = \mathfrak{b}_\alpha^m$ for some ideal $\mathfrak{b}_\alpha$

$$1 \to \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^m \to T_\phi \to \underbrace{I_K/K^\times}_{Cl_K} \quad \text{finite}$$

finite by Dirichlet

$$\alpha \longmapsto [\mathfrak{b}_\alpha]$$

$[\mathfrak{b}_\alpha] = 1 \iff \mathfrak{b}_\alpha = \beta\mathcal{O}_K \iff \alpha\mathcal{O}_K = \mathfrak{b}_\alpha^m = \beta^m\mathcal{O}_K \iff$

$\alpha = u\beta$ for some $u \in \mathcal{O}_K^\times$

hence T_ϕ finite.

$$0 \to E(K)/mE(K) \to H^1(G_{L/K}, E[m]) \to H^1(G_{\bar{K}/K}, E(\bar{K}))$$

L/K finite

$$0 \to E(K)/mE(K) \to \underbrace{S^{(m)}(E/K)}_{\text{selmer group}} \to \underbrace{Ш(E/K)[m]}_{\text{Shafarevich–Tate group}} \to 0$$

CM: describe abelian extensions of imaginary quadratic fields using torsion points on elliptic curves.

$K = \mathbb{Q}(\sqrt{-D})$, construct E/K with $\mathrm{End}(E) = \mathcal{O}_K$

Then $K(j(E)) =$ Hilbert class field of K.

Lecture 25 (2011-04-04)

We've proven weak Mordell-Weil / 2011-4-4
$E(\mathbb{Q})/mE(\mathbb{Q})$ is finite (25)

How does this get us that $E(\mathbb{Q})$ is finitely generated?
Define a "size" function and show $size(mP) > size(P)$

Height function — arithmetic complexity

Def. The height of $\alpha = \frac{a}{b} \in \mathbb{Q}$ is $H(\alpha) = \max\{|a|, |b|\}$

$P \in \mathbb{P}^n(\mathbb{Q})$, $P = [a_0, \ldots, a_n]$ where $a_i \in \mathbb{Z}$, $\gcd(a_i) = 1$

Then $H(P) = \max |a_i|$.

Thm. a) $\{P \in \mathbb{P}^n(\mathbb{Q}) \mid H(P) \le B\}$ is finite
b) Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be a degree d homogeneous map. Then $c_1 H(P)^d \le H(\phi(P)) \le c_2 H(P)^d$ where the c_i depend only on ϕ

H/W 1) What is $|\{P \in \mathbb{P}^n(\mathbb{Q}) \mid H(P) \le B\}|$?
2) Prove b.

$K/\mathbb{Q}$, $P = [a_0, \ldots, a_n] \in \mathbb{P}^n(K)$

$$H(P) = \prod_{v \in M_K} \max\{|a_0|_v, \ldots, |a_n|_v\}^{\frac{[K_v : \mathbb{Q}_v]}{[K:\mathbb{Q}]}}$$

this term does two things - ensures that the same point looked at in $\mathbb{P}^n(L)$ L/K, gives same height, and makes it well-defined - scaling P by λ introduces $\left(\prod_{v \in M_K} |\lambda|_v^{[K_v:\mathbb{Q}_v]}\right)^{1/[K:\mathbb{Q}]}$ $= 1$ by

HW: Check that definitions are the same when $K = \mathbb{Q}$.

Kronecker's Lemma: $\prod_{\alpha \in \mathbb{Q}} H(\alpha) = 1 \iff \alpha$ is a root of unity

(Analog for torsion points?)

$E: y^2 = x^3 + Ax + B$, $P = (x, y) \in E(K)$

Def: $H(P) = \begin{cases} H(x) & \text{if } P \neq \mathcal{O} \\ 1 & \text{if } P = \mathcal{O} \end{cases}$

$h(P) = \log(H(P))$

Prop: a) $\{P \in E(K) \mid h(P) \leq C\}$ is finite

b) $h(mP) = m^2 h(P) + \mathcal{O}_{E,m}(1)$

c) $h(P+Q) \leq 2h(P) + 2h(Q) + \mathcal{O}_E(1)$

Pf of Mordell-Weil Theorem: Choose $Q_1, \ldots, Q_r \in E(K)$ which represent the finitely many cosets of $E(K)/mE(K)$

Goal: Find some C such that

$\{Q_1, \ldots, Q_r\} \cup \{P \in E(K) \mid h(P) \leq C\}$

generates $E(K)$. Take any $P \in E(K)$, then we can write

$P = P_0 = mP_1 + Q_{i_1}$

$P_1 = mP_2 + Q_{i_2}$

$\vdots$

$P_{t-1} = mP_t + Q_{i_t}$

Some middle step: $P_{j-1} = mP_j + Q_{ij}$

$$h(mP_j) = h(P_{j-1} - Q_{ij}) \le 2h(P_{j-1}) + 2h(Q_{ij}) + C_1$$

$$\le 2h(P_{j-1}) + C_2 \quad \text{where } C_2 = 2\max\{h(Q_i)\} + C_1$$

but $h(mP_j) \ge m^2 h(P_j) - C_3$, so

$$h(P_j) \le \frac{2}{m^2} h(P_{j-1}) + C_4, \quad C_4 \text{ depends on } E, m, Q_1, \dots, Q_r$$

(Statement that $E(\mathbb{Q})/mE(\mathbb{Q})$ is finite for $m=1$ doesn't convey any info since $m \ge 2$)

Hence

$$h(P_t) \le \frac{2}{m^2} h(P_{t-1}) + C_4$$

$$\le \left(\frac{2}{m^2}\right)^2 h(P_{t-2}) + \left(\frac{2}{m^2} + 1\right) C_4$$

$$\le \left(\frac{2}{m^2}\right)^t h(P_0) + \left(\left(\frac{2}{m^2}\right)^{t-1} + \cdots + \left(\frac{2}{m^2}\right) + 1\right) C_4$$

$$\le \left(\frac{2}{m^2}\right)^t h(P_0) + \frac{1}{1 - \frac{2}{m^2}} C_4$$

$$\le \frac{1}{2^t} h(P_0) + 2C_4$$

We had $P = m^t P_t + \sum_{j=1}^{t} m^{j-1} Q_{ij}$, done.

How to make effective? need to find the $Q_1, \dots, Q_r$ (easy to find $\{P \in E(K) \mid h(P) \le C\}$, in principle). But $E(K)/mE(K) \hookrightarrow$ finite group (we know we showed $E(K)/mE(K)$ finite, and we don't know how to determine if an element of (finite group) came from $E(K)/mE(K)$)

$\alpha \in \overline{\mathbb{Q}}^\times$, then $h(\alpha^d) = d\, h(\alpha)$

$P \in E(K)$, then $h(mP) = m^2 h(P) + O_{E,m}(1)$

Thm (Neron–Tate) canonical height

$$\hat{h}(P) = \lim_{m\to\infty} \frac{1}{m^2} h(mP) \quad \text{exists}$$

$\lim_{m\to\infty} \frac{1}{m^3} h(mP)$ exists just always 0

and a) $\hat{h}(P) = h(P) + O_E(1)$

b) $\hat{h}(mP) = m^2 \hat{h}(P)$

The map $E(K) \times E(K) \to \mathbb{R}$

$$(P, Q) \longmapsto \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$$

is a bilinear form, non degenerate on $E(K)/$torsion

d) $\hat{h}(P) = 0 \iff P$ torsion pt. (analog of Kronecker's theorem)

Lehmer's conjecture

$\alpha \in \overline{\mathbb{Q}}^\times$, α not a root of unity

$$h(\alpha) \geq \frac{c}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$$

(c absolute constant (avoids silly $h(\sqrt{2}) = \frac{1}{2} h(2)$))

Elliptic version

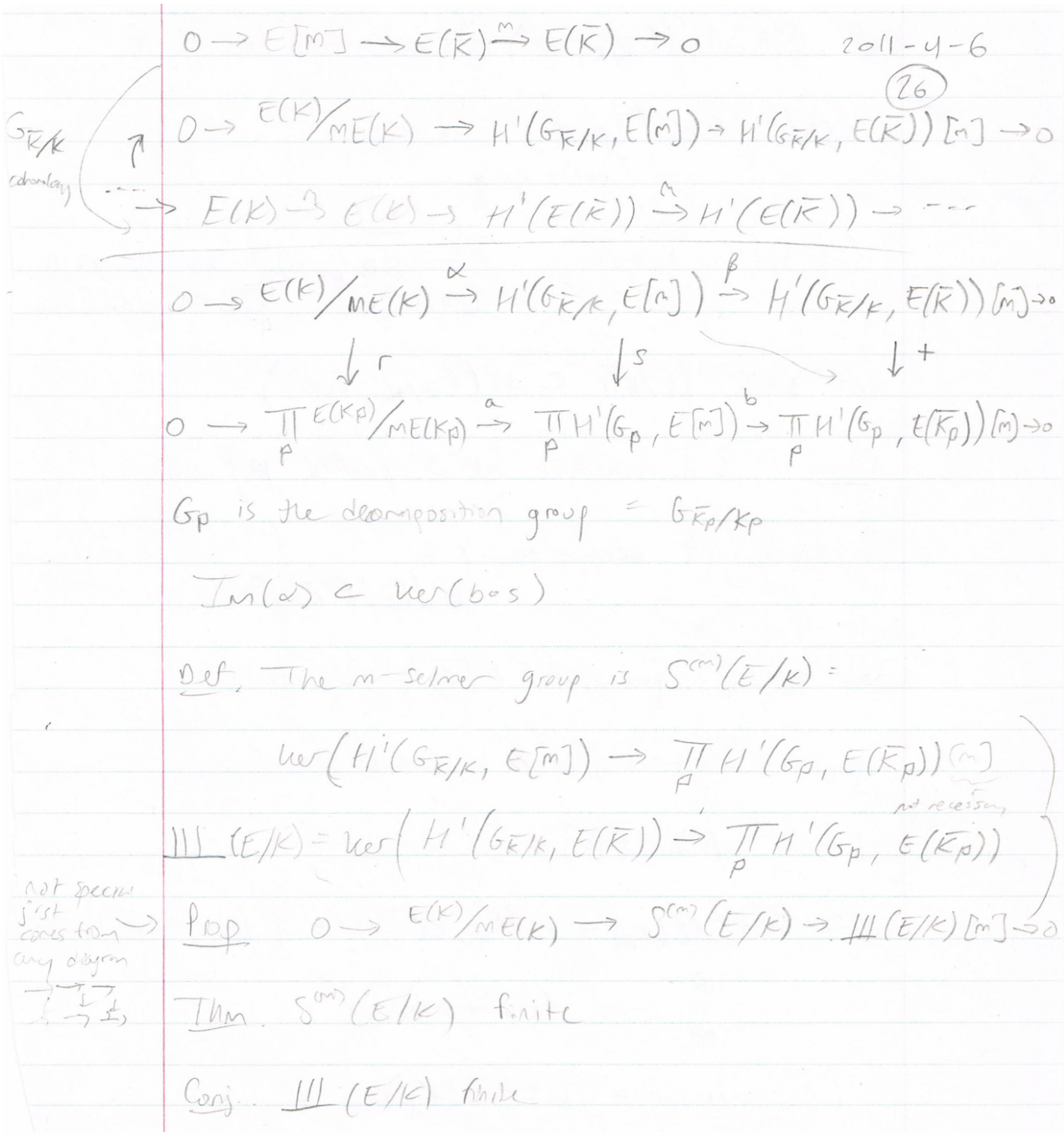
$$\hat{h}(P) \geq \frac{c_{E/K}}{[K(P):K]}$$

Lang's Conjecture

$E/\mathbb{Q}$, $P \in E(\mathbb{Q})$ non torsion

$\hat{h}(P) \geq c_1 \log|\mathrm{Disc}_E| - c_2$

Lecture 26 (2011-04-06)



Def $\xi \in H^1(G_{\bar{K}/K}, M)$ is unramified at p if

$\mathrm{Res}(\xi) \in H^1(I_p, M)$ is 0

I inertia group

Once fix an embedding $\bar{K} \hookrightarrow \bar{K_p}$, get specific group $I_p \subset G_p \subset G_{\bar{K}/K}$

Let $\xi \in S^{(m)}(E/K) \subset H^1(G_{\bar{K}/K}, E[m])$

Claim: ξ is unramified at p provided $p \nmid m\Delta_E$

↑ i.e., E stays an elliptic curve mod p

Pf We know ξ becomes trivial in $H^1(G_p, E(\bar{K_p}))[m]$

so there is a point $P \in E(\bar{K_p})$ such that

$\xi(\sigma) = \sigma(P) - P$ for all $\sigma \in G_p$

Notice $\xi(\sigma) \in E[m]$

$$0 \to E(K_p)/mE(K_p) \to H^1(G_p, E[m]) \to H^1(G_p, E(\bar{K_p}))[m] \to 0$$

$$R \quad \longrightarrow \quad (\sigma \mapsto \sigma(P) - P)$$

$\| \; mP$

since with $mP \in E(K_p)$

Now take $\tau \in I_p$, so τ fixes everything mod $\mathfrak{p}$

$\widetilde{\zeta(\tau)} = \widetilde{\tau(P) - P} \bmod \mathfrak{p} = \bar{\sigma} \bmod \mathfrak{p}$

$E[m] \hookrightarrow \widetilde{E[m]} \bmod \mathfrak{p}$

provided $p \nmid m$ and E has good reduction at p

$\therefore \zeta(\tau) = 0 \quad \forall \tau \in I_p$, hence ζ is unramified at p

Def M a finite $G_{\bar{K}/K}$-module
S finite set of primes

(e.g. $M = E[m]$, $S = \{p \mid m\Delta_E\}$)

$H^1(G_{\bar{K}/K}, M; S) = \{\zeta \in H^1(G_{\bar{K}/K}, M) \mid \zeta$ is unramified for all $p \notin S\}$

Prop. E/K, $S = \{p \mid m\Delta_E\}$. Then

$$S^{(m)}(E/K) \subset H^1(G_{\bar{K}/K}, E[m]; S)$$

Thm. $H^1(G_{\bar{K}/K}, M; S)$ is finite.

Pf ① Using inflation-restriction, can assume $G_{\bar{K}/K}$ acts trivially on M

② Let $m = \#M$, and adjoin $\mu_m \subset K$.

Let L = max abelian extension of K of exponent m that is unramified at every $p \notin S$

we have proved L/K finite

$$H^1(G_{\bar{K}/K}, M; S) \to \text{Hom}(G_{\bar{K}/K}, M; S) \hookrightarrow \text{Hom}(G_{L/K}, M)$$

finite

Example. $y^2 = (x-\alpha)(x-\beta)(x-\gamma)$, $\alpha, \beta, \gamma \in \mathbb{Z}$

$$S = \{2\} \cup \{p \mid (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)\}$$

Then $E(\mathbb{Q})/2E(\mathbb{Q}) \hookrightarrow S^{(2)}(E/\mathbb{Q}) \hookrightarrow (\mathbb{Q}^\times/\mathbb{Q}^{\times 2} \times \mathbb{Q}^\times/\mathbb{Q}^{\times 2})^S$

$\left(\langle -1, 17 \rangle / \text{squares}\right)^2$

$\text{rank}_{\mathbb{F}_2} E(\mathbb{Q})/2E(\mathbb{Q}) \leq 2(\#S+1)$

$\text{rank}\, E(\mathbb{Q}) \leq 2\#S$

$y^2 = x^3 - 12x^2 + 20x = x(x-2)(x-10)$

$S = \{-1, 2, 5\}$ rank $\in 6$

(actually rank $= 1$)

BSD conjecture

a) $\text{ord}_{s=1} L(E,s) = \text{rank}\, E(\mathbb{Q})$

$$L(E,s) = \prod_{p \nmid \Delta_E} \frac{1}{1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}} \cdot \prod_{p \mid \Delta_E} \frac{1}{1 - \frac{a_p}{p^s}}$$

b) $L(E,s) = c_E (s-1)^{\operatorname{rank} E(\mathbb{Q})} + \cdots$

$\Omega_E = \int_{E(\mathbb{R})} \frac{dx}{2y}$ $\qquad c_p = \#\left(E(\mathbb{Q}_p)/E_{ns}(\mathbb{Q}_p)\right)$

Γ_{pts} s.t. $\tilde{P}$ (mod p) is nonsingular

$\langle P, Q \rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$

Choose basis $P_1, \ldots, P_r$ for $E(\mathbb{Q})/E(\mathbb{Q})_{tors}$

$R_E = \det(\langle P_i, P_j \rangle)$

Conjecture

$$c_E = \Omega_E \left(\prod_{p \mid \Delta_E} c_p \right) \frac{R_E \, \#Ш(E/\mathbb{Q})}{(\#E(\mathbb{Q})_{tors})^2}$$

Lecture 27 (2011-04-08)

Elliptic Curves over $\mathbb{C}$ — 2011-4-8 (27)

Trig functions, $f(z+\omega) = f(z) \quad \forall z \in \mathbb{C}$

Def. A lattice in $\mathbb{C}$ is a subgroup of the form

$L = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}$ for ω_1, ω_2 $\mathbb{R}$-linearly independent

Def. An elliptic function for L is a meromorphic function $f: \mathbb{C} \to \mathbb{C}$ such that $f(z+\omega) = f(z) \quad \forall z \in \mathbb{C}, \omega \in L$

Def. A fundamental domain for $\mathbb{C}/L$ is

$D = \{a + t_1\omega_1 + t_2\omega_2 \mid 0 \le t_1, t_2 < 1\} \xleftrightarrow{\text{bijection}} \mathbb{C}/L$

not a homeomorphism

Let $\overline{D}$ = closure of D

Thm. Let $f \in \mathbb{C}(\Lambda)$. If f has no poles, then f is constant. If f has no zeros, then f is constant

Pf f continuous on $\overline{D}$, hence bounded on $\overline{D}$, hence bounded on $\mathbb{C}$, hence by Liouville $\Rightarrow$ f constant (use $1/f$ for second part).

Given $f \in \mathbb{C}(L)$ and $\omega \in \mathbb{C}$,

$\operatorname{ord}_\omega f$ = order of zero (pole) at ω

$\operatorname{res}_\omega f$ = residue at ω

$$f(z) = (z-w)^{\operatorname{ord}_w(f)} g(z)$$

$$f(z) = \cdots + \frac{\operatorname{res}_w(f)}{z-w} + \cdots$$

$$\operatorname{res}_w(f) = \frac{1}{2\pi i} \int_{|z-w|=\varepsilon} \frac{f(z)}{z-w}\, dz$$

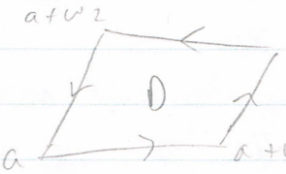
Prop. $f \in \mathbb{C}(L)$, then

a) $\sum_{w \in \mathbb{C}/L} \operatorname{ord}_w(f) = 0$

b) $\sum_{w \in \mathbb{C}/L} \operatorname{res}_w(f) = 0$

c) $\sum_{w \in \mathbb{C}/L} \operatorname{ord}_w(f) \cdot w =$

Pf. b) Cauchy Residue Thm, $\sum_{w \in R} \operatorname{res}_w(f) = \frac{1}{2\pi i} \int_{\partial R} f(z)\, dz$



(might have to move boundary)

$$2\pi i \sum_{w \in \mathbb{C}/L} \operatorname{res}_w(f) = \int_a^{a+w_1} + \int_{a+w_1}^{a+w_1+w_2} + \int_{a+w_1+w_2}^{a+w_2} + \int_{a+w_2}^{a}$$

$$= 0$$

a) $\operatorname{ord}_w f = \operatorname{res}_w \frac{f'}{f} = \operatorname{res}_w \frac{d}{dz} \log(f(z))$

$f'/f \in \mathbb{C}(L)$, use b) $\Rightarrow$ equals 0

c) HW. $\int_{\partial D} z \frac{f'(z)}{f(z)} dz$ look in book

What are some elliptic functions?

$\sum_{w \in L} \frac{1}{(z+w)^2}$? Doesn't quite converge

$f \in \mathbb{C}(L)$ f not constant $\Rightarrow$ # of poles ≥ 1

If # poles $= 1$, say $\operatorname{ord}_0 f = -1$

$\Rightarrow$ # zeros $= 1$ by a) but then by c) $z - t \in L$ Leave fingers in same contradiction

$\operatorname{ord}_t(f) = 1$

thus at least two poles

Weierstrass: double poles

Jacobi: poles at a and $-a$

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w \in L \\ w \neq 0}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

Thm. a) $\wp$ is absolutely convergent on all compact subsets of $\mathbb{C} - L$

b) $\wp(z) = \wp(-z)$ c) $\wp(z) \in \mathbb{C}(L)$

Pf b) $\frac{1}{(-z-w)^2} = \frac{1}{(z+w)^2}$

c) $\wp'(z) = -\frac{2}{z^3} + \sum_{w \neq 0} \frac{2}{(z-w)^3} = -2\sum_{w \in L} \frac{1}{(z-w)^3}$

clearly $\wp'(z) \in \mathbb{C}(L)$

so $\wp(z+w) = \wp(z) + c_w$ set $z = -\frac{w}{2}$

$\wp(\frac{w}{2}) = \wp(-\frac{w}{2}) + c_w$

equal

hence $c_w = 0$

Lecture 28 (2011-04-11)

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

2011-4-11

(28)

want to prove it is absolutely convergent on $\mathbb{C} - L$

Fix a $z \in \mathbb{C} - L$, suffices to show

$$\sum_{\substack{\omega \in L \\ |\omega| > B}} \left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \quad \text{finite for some } B$$

($B = 2|z|+1$ will work)

$$= \sum_{|\omega|>B} \frac{1}{|\omega|^2} \left| \frac{1}{(1-\frac{z}{\omega})^2} - 1 \right|$$

$$= \sum_{|\omega|>B} \frac{1}{|\omega|^2} \left| \sum_{n=1}^{\infty} n \left(\frac{z}{\omega}\right)^{n-1} - 1 \right|$$

$$= \sum_{|\omega|>B} \frac{1}{|\omega|^2} \left| \sum_{n=2}^{\infty} n \left(\frac{z}{\omega}\right)^{n-1} \right|$$

$$= \sum_{n=2}^{\infty} n |z|^{n-1} \sum_{|\omega|>B} \frac{1}{|\omega|^{n+1}}$$

$$\leq \sum_{n=2}^{\infty} n |z|^{n-1} \sum_{k=B}^{\infty} \frac{\#\{\omega \in L \mid k \leq |\omega| < k+1\}}{k^{n+1}}$$

$$\leq \sum_{n=2}^{\infty} n |z|^{n-1} \sum_{k=B}^{\infty} \frac{c_L \cdot \text{Area}\{s \in \mathbb{C} \mid k \leq |s| < k+1\}}{k^{n+1}}$$

$$\leq \sum_{n=2}^{\infty} n |z|^{n-1} \sum_{k=B}^{\infty} \frac{c_L \pi (2k+1)}{k^{n+1}} \leq 3 c_L \pi \underbrace{\sum_{n=2}^{\infty} n |z|^{n+1} \sum_{k=B}^{\infty} \frac{1}{k^n}}_{\text{converges}}$$

HW: $\#\{\omega \in L \mid |\omega| \le B\} = \frac{\pi B^2}{\text{Area}(\mathcal{O}_L)} + O(B)$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$= \frac{1}{z^2} + \sum_{\omega \in L}{}' \frac{1}{\omega^2} \left(\frac{1}{(1-\frac{z}{\omega})^2} - 1 \right)$$

$$= \frac{1}{z^2} + \sum_{\omega \in L}{}' \frac{1}{\omega^2} \sum_{n=2}^{\infty} n \left(\frac{z}{\omega}\right)^{n-1}$$

$$= \frac{1}{z^2} + \sum_{n=2}^{\infty} n z^{n-1} \left(\sum_{\omega \in L}{}' \frac{1}{\omega^{n+1}} \right)$$

Def. $G_n(L) = \sum_{\omega \in L}{}' \frac{1}{\omega^n}$. (Eisenstein series)

Prop. a) $G_n(L)$ absolutely convergent for $n \ge 3$ (add annuli)

b) n odd $\Rightarrow G_n(L) = 0$ (since $\omega \in L \iff -\omega \in L$)

Thm. In a neighborhood of 0,

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} n G_{n+1}(L) \cdot z^{n-1}$$

$$= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2}(L) \, z^{2k}$$

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots$$ — hint to cancel their poles

$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20 G_6 z^3 + \cdots$$

$$\wp'(z)^2 - 4\wp(z)^3 = (-24G_4\tfrac{1}{z^2} - 80G_6 + \cdots) - 4(9G_4\tfrac{1}{z^2} + 15G_6 + \cdots)$$

$$= -60G_4\tfrac{1}{z^2} - 140G_6 + \cdots$$

$$\underbrace{\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6}_{\text{meromorphic, elliptic (i.e. in } \mathbb{C}(L))} = \underbrace{0z^2 + 0z^4 + \cdots}_{\text{holomorphic, vanishes at } z=0}$$

$\downarrow$ LHS is constant 0

Thus

$$\wp'(z)^2 = 4\wp(z)^3 - \underbrace{60G_4(L)}_{g_2(L)}\wp(z) - \underbrace{140G_6(L)}_{g_3(L)}$$

Theorem: a) $4x^3 - g_2(L)x - g_3(L)$ has distinct roots

b) The map $\phi: \mathbb{C}/L \longrightarrow \mathbb{P}^2(\mathbb{C})$

$z \longmapsto [\wp(z), \wp'(z), 1]$

is a holomorphic isomorphism (as Riemann surfaces) from $\mathbb{C}/L$ to $E(\mathbb{C}): y^2 = 4x^3 - g_2x - g_3$

c) ϕ is a group homomorphism

Pf. a) $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ because $\wp$ even $\Rightarrow \wp'$ odd

$\swarrow$

$$\wp'(\tfrac{\omega_1}{2}) = -\wp'(-\tfrac{\omega_1}{2}) = -\wp'(\tfrac{\omega_1}{2} - \omega_1) = -\wp'(\tfrac{\omega_1}{2})$$

hence $\wp'(\frac{\omega_1}{2}) = 0$, similarly $\wp'(\frac{\omega_2}{2}) = 0$, also $\wp'(\frac{\omega_1+\omega_2}{2}) = 0$

Let $\omega_3 = \omega_1 + \omega_2$

$\wp(\frac{\omega_i}{2})$ is a root of $4x^3 - g_2 x - g_3$ $1 \le i \le 3$

$\wp(z) - \wp(\frac{\omega_1}{2})$ — zero at $\frac{\omega_1}{2}$
— derivative also zero at $\frac{\omega_1}{2}$
exactly 2 poles (since $\wp$ has 2 poles)
hence exactly 2 zeros on $\mathbb{C}/L$

thus there are no other solutions to $\wp(z) = \wp(\frac{\omega_1}{2})$ other than $z = \frac{\omega_1}{2}$

thus $4x^3 - g_2 x - g_2 = 4\prod_{i=1}^{3}(x - \wp(\frac{\omega_i}{2}))$
distinct

b) Φ is injective

Suppose $\wp(z_1) = \wp(z_2)$ and $\wp'(z_1) = \wp'(z_2)$

Can assume $2z_1 \notin L$ first.

$\wp(z) - \wp(z_1)$ — zero at $z = z_1, z = -z_1$
— zero at $z = z_2$
2 poles $\Rightarrow$ 2 zeros

thus $z_2 \equiv \pm z_1 \bmod L$

If $z_2 \equiv -z_1 \bmod L$, then $\wp'(z_2) = \wp'(-z_1) = -\wp'(z_1)$
contradiction

If $2z_1 \in L$, then —

ϕ surjective: Let $(a,b) \in E(\mathbb{C})$

$\wp(z) - a$ elliptic, 2 poles $\Rightarrow$ it has a zero

So $\exists\, z_0$ with $\wp(z_0) = a$. Then $\wp'(z_0)^2 = b^2$,

so $\pm z_0$ works to get $\phi(z_0) = (a,b)$

Lecture 29 (2011-04-13)

$\phi : \mathbb{C}/L \to E(\mathbb{C})$
$z \longmapsto (\wp(z), \wp'(z))$

2011-4-13
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Prop. ϕ is a homomorphism.

We need two facts: $\mathbb{C}(L) = \mathbb{C}(\wp(z), \wp'(z))$ (I)

(II) Let $D = \sum_w n_w w \in Div(\mathbb{C}/L)$, note
$div(f) = \sum ord_w(f)\, w$. We showed $div(f) \in L$.
In fact, $D = div(f) \Leftrightarrow D \in L$.

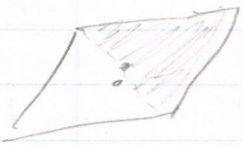
Pf. (I) Let $f \in \mathbb{C}(L)$

$f(z) + f(-z)$
$(f(z) - f(-z))\wp'(z)$
$\Big\}$ even elliptic functions

To show: $f \in \mathbb{C}(L)$, f even $\Rightarrow f \in \mathbb{C}(\wp(z))$

$$div(f) = \sum \underbrace{ord_w(f)}_{n_w} w = n_0(0) + \sum_{(\mathbb{C}/L)/(\pm 1)} n_w(w) + n_w(-w)$$

$$= n_0(0) + \sum_{(\mathbb{C}/L)/(\pm 1)} n_w((w) + (-w))$$



$div(\wp(z) - \wp(w)) = (w) + (-w) - 2(0)$

$g(z) = \prod_{w \in (\mathbb{C}/L)/(\pm 1)} (\wp(z) - \wp(w))^{n_w}$

$div(f(z)/g(z)) = N(0)$

$\Downarrow$

$div(f(z)/g(z)) = 0$

$div(g(z)) = n(0) + \sum n_w((w) + (-w))$

$f(z) = c \cdot g(z)$

$1 \to \mathbb{C}^\times \to \mathbb{C}(L)^\times \to Div^\circ(\mathbb{C}/L) \xrightarrow{\text{sum}} \mathbb{C}/L \to 0$

divisors of degree 0 ; $\mathbb{C}/L \cong E(\mathbb{C})$

$f \mapsto div(f)$

$Pic^\circ(\mathbb{C}/L) \overset{\text{def}}{=} Div^\circ(\mathbb{C}/L) / div(\mathbb{C}(L)^\times)$

$Pic^\circ : Var \to Grp$ (smooth projective $\to$ Grp.Var)

$V_1 \to V_2$ gives $Pic^\circ(V_1) \to Pic^\circ(V_2)$

$\mathbb{C}/L \longleftrightarrow E_L$

Maps between elliptic curves

Def. An *isogeny* $\phi : E_1 \to E_2$ is a morphism with $\phi(\mathcal{O}_1) = \mathcal{O}_2$

Thm. Every isogeny is a homomorphism

Prop. Let $\phi : \mathbb{C}/L_1 \to \mathbb{C}/L_2$ be holomorphic and $\phi(0) = 0$. Then $\exists \alpha \in \mathbb{C}^\times$ with $\alpha L_1 \subset L_2$, and $\phi(z) = \alpha z$ (i.e. $\phi(z \bmod L_1) = \alpha z \bmod L_2$)

Pf.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\psi} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/L_1 & \xrightarrow{\phi} & \mathbb{C}/L_2 \end{array}$$

$\psi(0) = 0$, ψ holomorphic (universal cover property)

Let $w \in L_1$, then

$\psi(z+w) - \psi(z) \in L_2$ So we get a map

$\mathbb{C} \to L_2$ (discrete) $\Rightarrow$

$z \mapsto \psi(z+w) - \psi(z)$

$\psi(z+\omega) = \psi(z) + c_\omega$ for all z

$\psi'(z+\omega) = \psi'(z)$ for all z and $\omega \in L_1$

$\psi' \in \mathbb{C}(L_1)$

but ψ holomorphic $\Rightarrow$ ψ' holomorphic

$\psi'(z) = \alpha$ for some $\alpha \in \mathbb{C}$

so $\psi(z) = \alpha z + \beta$, $\psi(0) = 0$

$\psi(z) = \alpha z$

$\phi(z) = \alpha z$

Thm 1 $\{\alpha \in \mathbb{C} \mid \alpha L_1 \subset L_2\} \longrightarrow \{$ holomorphic $\phi : \mathbb{C}/L_1 \to \mathbb{C}/L_2$, $\phi(0) = 0\}$

$\alpha \longmapsto (\phi_\alpha : z \mapsto \alpha z)$

is bijective

Pf. we just did onto. injective: $\phi_\alpha = \phi_\beta \Rightarrow$

$(\alpha - \beta) z \in L_1 \quad \forall z \in \mathbb{C}$

$\Rightarrow \alpha = \beta$

Thm 2. $\{$isogenies $E_1 \to E_2\} \to \{$ holomorphic $\phi : \mathbb{C}/L_1 \to \mathbb{C}/L_2$, $\phi(0) = 0\}$

Group $\mathrm{Hom}(E_1, E_2) \subset \mathbb{C}$

Ring $\mathrm{End}(E_1) \subset \mathbb{C}$

Lecture 30 (2011-04-15)

2011-4-15 (30)

(a) Thm. $\{\alpha \in \mathbb{C} : \alpha L_1 \subset L_2\} \xrightarrow{\text{bijective}} \{\text{holomorphic } \phi : \mathbb{C}/L_1 \to \mathbb{C}/L_2 \mid \phi(0)=0\}$

$\alpha \longmapsto \phi_\alpha(z) = \alpha z$

(b) $\{\text{isogenies } E_{L_1} \to E_{L_2}\} \xrightarrow{\text{bijective}} \{\text{holomorphic } \mathbb{C}/L_1 \to \mathbb{C}/L_2 \mid 0 \mapsto 0\}$

Pf. (a) last time,

(b) well-defined, injective

$$\begin{array}{ccc} E_{L_1}(\mathbb{C}) & \longrightarrow & E_{L_2}(\mathbb{C}) \\ \| & & \| \\ \mathbb{C}/L_1 & \longrightarrow & \mathbb{C}/L_2 \end{array}$$

Let $\phi_\alpha : \mathbb{C}/L_1 \to \mathbb{C}/L_2$ $\alpha L_1 \subset L_2$

Get a map $E_{L_1}(\mathbb{C}) \to E_{L_2}(\mathbb{C})$

$$(\wp(z, L_1), \wp'(z, L_1)) \longmapsto (\wp(\alpha z, L_2), \wp'(\alpha z, L_2))$$

Need to show $\wp(\alpha z, L_2), \wp'(\alpha z, L_2) \in \mathbb{C}(\wp(z, L_1), \wp'(z, L_1))$

Let $w \in L_1$ then (last time: $= \mathbb{C}(L_1)$)

$$\wp(\alpha(z+w), L_2) = \wp(\alpha z + \alpha w, L_2) = \wp(\alpha z, L_2) \quad \checkmark$$

$\alpha w \in L_2$

Thm. $V, W \subset \mathbb{P}^n(\mathbb{C})$ be smooth projective varieties. If $\phi : V(\mathbb{C}) \to W(\mathbb{C})$ is holomorphic, then it is algebraic.

Corollary. $E_{L_1} \cong E_{L_2} \iff \exists c \in \mathbb{C}^\times$ with $cL_1 = L_2$

Def. $\mathrm{End}(E) = \{\text{isogenies } E \to E\}$

$(\phi + \psi)(P) = \phi(P) + \psi(P)$

$(\phi\psi)(P) = \phi(\psi(P))$

$\mathbb{Z} \hookrightarrow \mathrm{End}(E)$

$n \mapsto (P \mapsto nP)$

Corollary. $\mathrm{End}(E_L) \cong \{\alpha \in \mathbb{C} \mid \alpha L \subset L\}$

Corollary. $E_L[n] = \ker\left(\mathbb{C}/L \xrightarrow{z \mapsto nz} \mathbb{C}/L\right) = \frac{1}{n}L/L$

If $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, this is $\cong \frac{1}{n}\mathbb{Z}/\mathbb{Z} \cdot \omega_1 + \frac{1}{n}\mathbb{Z}/\mathbb{Z} \cdot \omega_2$

Uniformization Thm. Let $E/\mathbb{C}$ be an elliptic curve

then $\exists! L$ such that $\mathbb{C}/L \to E(\mathbb{C})$

$(\wp, \wp')$

$g_2(L) = -a$

$g_3(L) = -b$

Thm. Let $E/\mathbb{C}$. Then

$\mathrm{End}(E) = \begin{cases} \mathbb{Z} \\ \text{order in a quadratic imaginary field} \end{cases}$ (CM)

Def. $K/\mathbb{Q}$ number field, an order is a subring $R \subset \mathcal{O}_K$ with $R \otimes_{\mathbb{Z}} \mathbb{Q} = K$.

Lecture 31 (2011-04-18)

$z \longmapsto (\wp(z), \wp'(z))$

$\mathbb{C}/L \longrightarrow E_L(\mathbb{C})$ 2011-4-18

(31)

$\mathrm{End}(E_L) \cong \mathrm{End}(L) = \{\alpha \in \mathbb{C} \mid \alpha L \subset L\}$

Invariant differential

$$\omega_L = \frac{dx}{y} = \frac{d\wp(z)}{\wp'(z)} = dz$$

Given $\alpha \in \mathrm{End}(L)$, get $[\alpha]_{E_L} \in \mathrm{End}(E_L)$, make precise by requiring $[\alpha]_{E_L}^* \omega_{E_L} = \alpha \omega_{E_L}$ (alternative would be $\bar{\alpha} \omega_{E_L}$)

$\phi : E_1 \to E_2 \qquad \phi \circ [\alpha]_{E_1} = [\alpha]_{E_2} \circ \phi$

K quadratic imaginary field, R ring of integers of K

$\mathcal{E}\ell\ell(R) = \{E/\mathbb{C} \mid \mathrm{End}(E) = R\} / \mathbb{C}\text{-isomorphism}$

$\cong \{\text{lattices } L \subset \mathbb{C} \mid \mathrm{End}(L) = R\} /$ homothety

Is there an $E/\mathbb{C}$ with $\mathrm{End}(E) = R$? Yes.

E_R (R is a lattice) $\quad E_R(\mathbb{C}) = \mathbb{C}/R$

If $\mathfrak{a}$ is any fractional ideal, $E_{\mathfrak{a}} \in \mathcal{E}\ell\ell(R)$

Get $Cl(K) \longrightarrow \mathcal{E}\ell\ell(R)$ (action, simply transitive)

$\mathfrak{a} \longmapsto E_{\mathfrak{a}}$

$y^2 = x^3 + x$, $R = \mathbb{Z}[i]$

$[i](x,y) = (-x, iy)$

$y^2 = x^3 + 1$ $R = \mathbb{Z}[\zeta_3]$

$[\zeta_3](x,y) = (\zeta_3 x, y)$

$y^2 = x^3 + 4x^2 + 2x$ $R = \mathbb{Z}[\sqrt{-2}]$

$[\sqrt{-2}](x,y) = \left(-\frac{1}{2}\left(x + 4 + \frac{2}{x}\right), -\frac{1}{2\sqrt{-2}}\, y\left(1 - \frac{2}{x^2}\right)\right)$

$\ker[\sqrt{-2}] = \{\mathcal{O}, (0,0)\}$

HW. Let $R \subset \mathbb{Q}(\sqrt{-D})$ be an order, prove $\exists! c \in \mathbb{N}$ with $R = \mathbb{Z} + c\mathcal{O}_K$ (c is the conductor)

Pf of Thm. $E = E_L$, $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$\mathrm{End}(E) = \{\alpha \in \mathbb{C} \mid \alpha L \subset L\}$

$$\alpha\omega_1 = a\omega_1 + b\omega_2$$
$$\alpha\omega_2 = c\omega_1 + d\omega_2$$

$$\begin{pmatrix} \alpha - a & -b \\ -c & \alpha - d \end{pmatrix} \underbrace{\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}}_{\text{not } 0} = 0$$

$$\det = \alpha^2 - (a+d)\alpha + (ad - bc) = 0$$

$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$, because $\alpha = a + b\frac{\omega_2}{\omega_1}$, $\frac{\omega_2}{\omega_1} \in \mathbb{C} - \mathbb{R}$, $\alpha \notin \mathbb{Z}$

so $\mathbb{Q}(\alpha) = \mathbb{Q}(\frac{\omega_2}{\omega_1})$, quadratic imaginary

$R = \mathrm{End}(E) \subset K$, $R \otimes \mathbb{Q} = K$ ⌐

If $\mathfrak{a} \subset K$ a fractional ideal; L a lattice

$$\mathfrak{a}L = \{a_1\lambda_1 + \cdots + a_n\lambda_n \mid a_i \in \mathfrak{a},\ \lambda_i \in L\}$$

Thm. a) $L \subset \mathbb{C}$ a lattice, with $\mathrm{End}(\mathbb{C}_L) = R$, $\mathfrak{a}$, $\mathfrak{b}$ fractional ideals

1) $\mathfrak{a}L$ is a lattice
2) $E_{\mathfrak{a}L} \in \mathcal{E}\ell\ell(R)$
3) $E_{\mathfrak{a}L} \cong E_{\mathfrak{b}L} \iff \bar{\mathfrak{a}} = \mathfrak{b}$ in $Cl(K)$

b) We get a well-defined action of $Cl(K)$ on $\mathcal{E}\ell\ell(R)$

$$\bar{\mathfrak{a}} * E_L = E_{\bar{\mathfrak{a}}^{-1}L}$$

and this action is simply transitive:

$$\bar{\mathfrak{a}} * E_L = \bar{\mathfrak{b}} E_L \iff \bar{\mathfrak{a}} = \bar{\mathfrak{b}}$$

So $\#Cl(K) = \#\mathcal{E}\ell\ell(R)$

Pf 1) We can find $d \in \mathbb{Z}$, $d \geq 1$ with $d\mathfrak{a} \subset R$ and $dR \subset \mathfrak{a}$, i.e. $dR \subset \mathfrak{a} \subset \frac{1}{d}R$. Since $\mathrm{End}(L) = R$, we have $RL = L$. Thus

$$\underbrace{dRL}_{=dL} \subset \mathfrak{a}L \subset \underbrace{\tfrac{1}{d}RL}_{=\frac{1}{d}L}$$

dL: spans; $\frac{1}{d}L$: discrete

2) $\mathrm{End}(E_{aL}) = \mathrm{End}(aL) = \{\alpha \in \mathbb{C} \mid \alpha aL \subset aL\}$ (mult by a^{-1})

$= \{\alpha \in \mathbb{C} \mid \alpha RL \subset RL\} = \{\alpha \in \mathbb{C} \mid \alpha L \subset L\} =$

$\mathrm{End}(L) = \mathrm{End}(E_L) = R$

3) ($\Leftarrow$) $\bar{a} = \bar{b} \Rightarrow a = cb \Rightarrow a, b$ homothetic

$\Rightarrow E_{aL} \cong E_{bL}$

($\Rightarrow$) $E_{aL} \cong E_{bL} \Rightarrow aL = cbL$ for some

$c \in \mathbb{C} \Rightarrow \begin{cases} ac^{-1}b^{-1}L = L \\ a^{-1}cbL = L \end{cases}$

$\mathrm{End}(L) = R$, so $\begin{matrix} ac^{-1}b^{-1} \subset R \\ a^{-1}cb \subset R \end{matrix} \Rightarrow c \in K \Rightarrow a = cb$

hence $\bar{a} = \bar{b}$

b) let $E_1, E_2 \in \mathcal{E}\ell\ell(R)$

$L_1 = \mathbb{Z} + \mathbb{Z}\tau_1$ for E_1

$L_1 = \mathbb{Z} + \mathbb{Z}\tau_2$ for E_2

From before, $\tau_1, \tau_2 \in K$ $(\mathbb{Q}(\tau_1) = \mathbb{Q}(\tau_2) = K)$

$L_1 = L_2$ are fractional ideals of K

$\| \quad \|$

$a_1 \quad a_2$ $\qquad E_1 = E_{L_1} = E_{a_1}$

$\overline{a_1 a_1^{-1}} * E_1 = \overline{a_1 a_2^{-1}} * E_{a_1} = E_{a_2} = E_2$

hence transitive

Simply transitive by 3.

Example. $L = \mathbb{Z}[i]$, $\mathrm{End}(E_L) = \mathbb{Z}[i]$

$E_L : y^2 = 4x^3 - g_2(L)x - g_3(L)$

$g_3(L) = g_3(iL) = i^{-6} g_3(L) = -g_3(L) \Rightarrow$

$g_3(L) = 0$

so $[i](x, y) = (-x, iy)$

(Check this is the right one – i.e., $\int \frac{dx}{y} = i \frac{dx}{y}$)

Thm Hurwitz. $g_2(\mathbb{Z}[i]) = 64 \left(\int_0^1 \frac{dt}{\sqrt{1-t^4}} \right)^4$

period

$E/\mathbb{C}$, $K/\mathbb{Q}$, $\mathrm{End}(E) = \mathcal{R}$

Goal #1 $K(j(E)) =$ Hilbert class field of K

Let $\sigma_{\mathfrak{p}} = (\mathfrak{p}, K(j(E))/K)$, write $E = E_{\mathfrak{L}}$

$\sigma_{\mathfrak{p}}(j(E_{\mathfrak{L}})) = j(\bar{\mathfrak{p}} * E_{\mathfrak{L}}) = j(E_{\mathfrak{p}^{-1}\mathfrak{L}})$

In general, $\sigma_{\mathfrak{p}}($analytic function of $\gamma) =$ analytic function of $\mathfrak{p}^{-1}\gamma$

Note that $\sigma \in \mathrm{Aut}(\mathbb{C})$, in general

$$\sigma\left(\sum_{n=0}^{\infty} a_n\right), \quad a_n \in \overline{\mathbb{Q}}$$
$$\neq \sum_{n=0}^{\infty} \sigma(a_n)$$

Lecture 32 (2011-04-20)

2011-4-20 (32)

Prop. a) $E/\mathbb{C}$, $\sigma \in \mathrm{Aut}(\mathbb{C})$
Then $\mathrm{End}(E^\sigma) \cong \mathrm{End}(E)$

b) If E has CM, then $j(E) \in \overline{\mathbb{Q}}$.

c) $\mathcal{E}\ell\ell(R) = \dfrac{\{E/\overline{\mathbb{Q}} : \mathrm{End}(E) = R\}}{\overline{\mathbb{Q}}\text{-isomorphism}}$

Pf. a) $\mathrm{End}(E) \xrightarrow{\sim} \mathrm{End}(E^\sigma)$ $\qquad E^\sigma = \sigma(E)$, $\phi^\sigma = \sigma(\phi)$
$\phi \longmapsto \phi^\sigma$
$\gamma^{\sigma^{-1}} \longmapsto \gamma$

b). Let $E \in \mathcal{E}\ell\ell(R)$ (really an equivalence class)

Then $\forall \sigma \in \mathrm{Aut}(\mathbb{C})$, $E^\sigma \in \mathcal{E}\ell\ell(R)$

$\dfrac{\{E^\sigma : \sigma \in \mathrm{Aut}(\mathbb{C})\}}{\text{isomorphism}} \subseteq \underbrace{\mathcal{E}\ell\ell(R)}_{\text{finite}}$ $\qquad \#\mathcal{E}\ell\ell(R) = \#Cl(R)$

$\{j(E^\sigma) : \sigma \in \mathrm{Aut}(\mathbb{C})\}$

$\{j(E)^\sigma : \sigma \in \mathrm{Aut}(\mathbb{C})\}$ $\qquad E^\sigma : y^2 = x^3 + A^\sigma x + B^\sigma$

finite

Hence $j(E)$ algebraic and $[\mathbb{Q}(j(E)) : \mathbb{Q}] \le h_K$

c) follows from a) + b)

$R \longrightarrow \mathrm{End}(E)$
$\alpha \longmapsto [\alpha]_E$

Thm. a) $E \in \mathcal{E}\ell\ell(R)$ $\alpha \in R$, $\sigma \in \mathrm{Aut}(\mathbb{C})$

$([\alpha]_E)^\sigma = [\alpha^\sigma]_{E^\sigma}$

b) $E \in \mathcal{E}\ell\ell(R)$, say E defined over some field L
$K = R \otimes_{\mathbb{Z}} \mathbb{Q}$

Then $\mathrm{End}(E) = \mathrm{End}_{LK}(E)$

Pf. a) $\omega_E = \frac{dx}{y}$ $\quad [\alpha]_E^* \omega_E = \alpha \omega_E$

$\omega_{E^\sigma} = \frac{dX}{Y}$

so $(\omega_E)^\sigma = \omega_{E^\sigma}$

$E: y^2 = x^3 + Ax + B$
$E^\sigma: Y^2 = X^3 + A^\sigma X + B^\sigma$

$(([\alpha]_E)^\sigma)^* \omega_{E^\sigma}^\sigma = \alpha^\sigma \omega_{E^\sigma}^\sigma$

but $(([\alpha]_E)^\sigma)^* \omega_{E^\sigma} = \alpha^\sigma \omega_{E^\sigma} = ([\alpha^\sigma]_{E^\sigma})^* \omega_{E^\sigma}$

$\mathrm{End}(E) \hookrightarrow \mathrm{End}(\text{differential forms})$
$\phi \longmapsto \phi^*$

because $\phi = [\alpha]_E$ for some α and $[\alpha]_E^* \omega_E = \alpha \omega_E$
so kernel $= \{\alpha = 0\}$

this because they have the same action,
$([\alpha]_E)^\sigma = [\alpha^\sigma]_{E^\sigma}$

b) $[\alpha] \in \mathrm{End}(E)$, where E/L

If σ fixes LK, then $E^\sigma = E$ and $\alpha^\sigma = \alpha$

So $([\alpha]_E)^\sigma = [\alpha^\sigma]_{E^\sigma} = [\alpha]_E$

so $[\alpha]_E$ fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/LK)$

Thm.

Corollary. $E \in \mathcal{E}\ell\ell(R)$, then $K(j(E), E_{tors})/K(j(E))$ is an abelian extension

Pf. Let $H = K(j(E))$, $L = H(E[m])$

Enough to show L/H is abelian. We have

$$\rho : G_{L/H} \hookrightarrow \mathrm{Aut}(E[m]) = GL_2(\mathbb{Z}/m\mathbb{Z})$$
$$\sigma \longmapsto (P \mapsto P^\sigma)$$

Notice that if $\phi \in R \cong \mathrm{End}(E)$, $\phi^\sigma = \phi$ $\forall \sigma \in G_{L/H}$

$$\rho(G_{L/H}) \subset \mathrm{Aut}_{R/mR}(E[m]) \cong (R/mR)^\times$$

($\mathrm{Aut}_{R/mR}$: anything R/mR-acts trivially)

$E[m] = \frac{1}{m}L/L$ and $RL = L$ because $E[m] \cong$ free R/mR-module of rank 1

$\rho : G_{L/H} \hookrightarrow (R/mR)^\times$

Thm. $\exists\ C_E$ with $[(R/mR)^\times : \rho(G_{L/H})] \leq C_E$

Thm (Serre). If $\exists$ doesn't have C_E, $\exists\ C_E$ with

$$[GL_2(\mathbb{Z}/m\mathbb{Z}) : \rho(G_{L/H})] \leq C_E$$

- $G_{\bar{K}/K}$ acts on $Cl(R)$
- $Cl(R)$ acts simply transitively on $Ell(R)$

So fix some $E \in Ell(R)$, get a map

$$F_E : G_{\bar{K}/K} \to Cl(R) \qquad \bar{\alpha}^* E_L \overset{\text{as}}{=} E_{\alpha^{-1}L}$$

$$\sigma \longmapsto F(\sigma) \quad \left(\begin{array}{c}\text{Unique} \\ \text{such that} \\ F(\sigma) * E = E^\sigma\end{array}\right)$$

Thm. F is a homomorphism, F is independent of E

Lecture 33 (2011-04-22)

$K/\mathbb{Q}$, $R = \mathcal{O}_K$, $E \in \mathcal{E}\ell\ell(R)$ 2011-4-22 (33)

$F: Gal(\bar{K}/K) \longrightarrow Cl(R)$ defined by

$\sigma \longmapsto F(\sigma)$ such that $F(\sigma) \cdot E = E^{\sigma}$

Thm. a) F is a homomorphism

b) F is independent of the choice of $\bar{E}$

Pf a) easy from the def

b) see book (key: show $(\bar{\alpha} * E)^{\sigma} = \bar{\alpha} * E^{\sigma}$

Thm. $E \in \mathcal{E}\ell\ell(R)$

a) $K(j(E)) = H$ Hilbert class field of K

b) $[K(j(E)) : K] = [\mathbb{Q}(j(E)) : \mathbb{Q}] = h$

c) $\{ j(E)^{\sigma} \mid \sigma \in G_{\bar{K}/K} \} = \{ j(E') \mid E' \in \mathcal{E}\ell\ell(R) \}$

d) $j(E)^{(\alpha, H/K)} = j(\bar{\alpha} * E)$

Lemma There is a finite set of primes $S \subset \mathbb{Z}$ such that for all $p \notin S$ that split as $p = \mathfrak{P}_1\mathfrak{P}_2$ in K,

$$F(\sigma_{\mathfrak{P}}) = \bar{p} \in Cl(R)$$

Frobenius, $(\mathfrak{P}, K^{ab}/K)$

Pf (Lemma ⇒ Thm) $F: G_{\bar{K}/K} \to Cl(R)$

Let L = fixed field of $\ker(F)$, so

$F: G_{L/K} \hookrightarrow Cl(R)$

$G_{\bar{K}/L} = \ker(F) = \{\sigma \in G_{\bar{K}/K} \mid F(\sigma) = 1\}$

$= \{\sigma \in G_{\bar{K}/K} \mid F(\sigma) * E \cong E\}$

$= \{\sigma \in G_{\bar{K}/K} \mid E^\sigma = E\} = \{\sigma \in G_{\bar{K}/K} \mid j(E^\sigma) = j(E)\}$

$= \{\sigma \in G_{\bar{K}/K} \mid j(E)^\sigma = j(E)\} = G_{\bar{K}/K(j(E))}$

so $L = K(j(E))$.

Claim $I(\mathfrak{C}_{L/K}) \xrightarrow{(\cdot, L/K)} G_{L/K} \xrightarrow{F} Cl(R)$ ($\mathfrak{C}_{L/K}$: conductor of L/K)

$F((\mathfrak{a}, L/K)) = \bar{\mathfrak{a}}$

Pf of Claim, use the fact that $I(\mathfrak{C}_{L/K})/P(\mathfrak{C}_{L/K})$ is finite and Dirichlet's thm (generally, Tchebotarev thm) guarantees each ideal class in contains infinitely many degree 1 prime ideals. So we can find $\mathfrak{p}$ a prime ideal (i.e. norm down to $\mathbb{Q}$ = prime)

with $\bar{a} = \bar{p}$ in $I(C_{L/K})/P(C_{L/K})$.

and $N_{K/\mathbb{Q}} \mathfrak{P} = p \notin S$ (by Lemma, infinitely many primes – finitely many = infinitely many)

$a = \beta p$ for some $\beta \equiv 1 \bmod C_{L/K}$

Lemma says $F((p, L/K)) = \bar{p} \in Cl(R)$

$(a, L/K) = (\beta p, L/K) = (p, L/K)$ because $\beta \equiv 1 \bmod C_{L/K}$

$F((a, L/K)) = \bar{\mathfrak{P}} = \bar{a}$ ✓

In particular, $F((\alpha R, L/K)) = \overline{\alpha R} = 1$ in $Cl(R)$

F is injective on $G_{L/K}$, so $(\alpha R, L/K) = 1$

for all (Calmostly) principal ideals αR with $(\alpha R, L/K) = 1$

this forces $G_{L/K} = (1)$ ("too many" things going to 1)

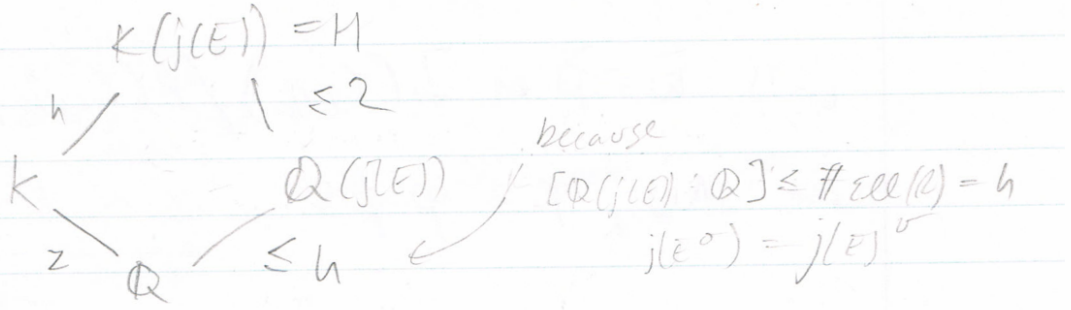
so L/K unramified, so $L \subset H$

but $I_K \twoheadrightarrow Cl(R)$ so $[L:K] \geq h$

$F((a, L/K)) = \bar{a}$

$F: G_{L/K} \twoheadrightarrow Cl(R)$ So $L = H$

this is a), and half of b)



So we must actually have $[\mathbb{Q}(j(E)) : \mathbb{Q}] = h$

c) follows because we know $Cl(R)$ acts transitively on $Ell(R)$ and $F : G_{L/K} \twoheadrightarrow Cl(R)$

$\Rightarrow G_{L/K}$ acts transitively on $Ell(R)$

d) claim says $F((a, L/K)) = a$

$\Rightarrow E^{(a, L/K)} = F((a, L/K)) * E = \bar{a} * E$

from claim, now take j's

$K = \mathbb{Q}(\sqrt{-D})$, $R = \mathbb{Z}[\tau]$, $\tau = \frac{D + \sqrt{-D}}{2}$

$g_2(D + \tau\mathbb{Z})$
$g_3(D + \tau\mathbb{Z})$

$j = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2}$

$G_k(D + \tau\mathbb{Z}) = \sum_{m,n \in \mathbb{C}}$

$q = e^{2\pi i \tau}$

$j(\tau) = \frac{1}{q} + \sum_{n \geq 0} c_n q^n \quad c_n \in \mathbb{Z}$

Lecture 34 (2011-04-25)

$I(\mathfrak{b})$ = fractional ideals prime to $\mathfrak{b}$ 2011-4-25

$P(\mathfrak{b}) = \{(\alpha) \mid \gcd((\alpha), \mathfrak{b}) = 1\}$ (34)

$P_1(\mathfrak{b}) = \{(\alpha) \mid \alpha \equiv 1 \bmod \mathfrak{b}\}$

lemma. $I(\mathfrak{b})/P(\mathfrak{b}) \xrightarrow{\sim} I(1)/P(1) = Cl(R)$

Pf. $I(\mathfrak{b}) \to I(1)/P(1)$

$\ker = I(\mathfrak{b}) \cap P(1) = P(\mathfrak{b})$, so injective

Let $\mathfrak{a} \in I(1)$, look at $\bar{\mathfrak{a}} \in Cl(R)$. By

Dirichlet, $\exists$ prime $\mathfrak{p} \in \bar{\mathfrak{a}}$ with $\mathfrak{p} \nmid \mathfrak{b}$ (only finitely many primes dividing $\mathfrak{b}$)

Then $\mathfrak{p} \in I(\mathfrak{b}) \leftrightarrow \bar{\mathfrak{p}} = \bar{\mathfrak{a}}$

Prop. L/K abelian, Suppose $P(C_{L/K}) \subseteq \ker(\cdot, L/K)$

Then $L \subset H$ = Hilbert class field.

Pf. Replace L by LH, note $C_{LH/K} = C_{L/K}$

Know $H \subset L$, so now want to show $H = L$.

$$\begin{array}{ccc} \frac{I(C_{L/K})}{P(C_{L/K})} & \xrightarrow{(\cdot, L/K)} & G_{L/K} \\ \text{lemma} \downarrow & & \downarrow \text{restriction} \\ Cl(R) = \frac{I(1)}{P(1)} & \xrightarrow[\sim]{(\cdot, H/K)} & G_{H/K} \end{array}$$

Thus

$$\#G_{H/K} = \#\,Cl(R) = \#\frac{I(C_{L/K})}{P(C_{L/K})} \geq \#G_{L/K} \geq \#G_{H/K}$$

Thus $G_{H/K} = G_{L/K}$; so $L = H$.

Topics to prove Lemma

Frobenius map and inseparability

$E: y^2 = x^3 + Ax + B \qquad E^{(p)}: y^2 = x^3 + A^p x + B^p$

The Frobenius map $\phi_p : E \to E^{(p)}$ is $(x, y) \mapsto (x^p, y^p)$

ϕ_p is purely inseparable:

Def. Let $\phi : C_1 \to C_2$ be a non-constant morphism between nonsingular projective curves

$$\phi^* : k(C_2) \to k(C_1)$$
$$f \longmapsto f \circ \phi$$

$$\deg(\phi) = [k(C_1) : \phi^* k(C_2)]$$

$\deg(\phi_p) = p \qquad \deg_i(\phi) =$ inseparability degree of $k(C_1)/\phi^* k(C_2)$

Prop. If $\phi : C_1 \to C_2$ is inseparable, then ϕ factors as $C_1 \xrightarrow{\varphi_p} C_1^{(q)} \xrightarrow{\psi} C_2$ with $\phi = \psi \circ \varphi_p$.

Pf. Given L/K, char $= p$, can write

L
| purely inseparable
M
| separable
K

How can we tell whether ϕ is inseparable?

$f(x)$ irreducible $\in K[x]$, $K(\alpha)/K$ inseparable $\iff$ $f(\alpha) = 0$, $f'(\alpha) = 0 \iff f'(x) = 0$

Let ω = differential 1-form on C_2

$\omega = f\,dg$ where $f, g \in k(C)$

$\phi : C_1 \to C_2$, ω on C_2

$\phi^* \omega = (\phi^* f)\, d(\phi^* g)$

Prop $\phi: C_1 \to C_2$ is inseparable iff

$\phi^* \omega = 0$, where $\omega \neq 0$ is a 1-form on C_2

Pf. ϕ insep $\iff \phi = \gamma \circ \phi_p$ some

$\gamma: C_1^{(p)} \to C_2$

Corollary. If $p \nmid n$ then $n + m\phi_p$ is separable

If. $(n + m\phi_p)^* \omega = n^* \omega + m\phi_p^* \omega$

$\neq 0$

Used to prove $\#E(\mathbb{F}_p) = p + 1 - a$, $|a| < 2\sqrt{p}$

$(p, 4/k)\ E = (\overline{p} + E)$ Frobenius

$E \longrightarrow \overline{p}^* E$ inseparable when reduce mod p

Reduction of isogenies mod $\mathfrak{p}$

good reduction: E/L, L number field

$\tilde{E}$ mod $\mathfrak{p}$: $y^2 = x^3 + \tilde{A}x + \tilde{B}x$

If $\phi: E \to E$, can reduce $\tilde{\phi}: \tilde{E} \to \tilde{E}$

Thm $\deg(\widehat{\phi}) = \deg(\phi)$

Pf idea. Look at action on torsion points

because $E[\ell] \to \widetilde{E}[\ell]$ is an isomorphism if $\beta \nmid \ell$.

$$\begin{array}{ccc} E[\ell] & \xrightarrow{\phi} & E[\ell] \\ \downarrow & & \downarrow \\ \widetilde{E}[\ell] & \xrightarrow[\widehat{\phi}]{} & \widetilde{E}[\ell] \end{array}$$

Lecture 35 (2011-05-04)

Main Thm. $j(E)^{(\mathfrak{a}, K^{ab}/K)} = j(\overline{\mathfrak{a}}^{*}E)$ 2011-5-4 (35)

Lemma. $\exists$ finite set of primes S such that if $p \notin S$, p splits as $pR = \mathfrak{P}\mathfrak{P}'$ in R

Then $j(E)^{(\mathfrak{P}, K^{ab}/K)} = j(\overline{\mathfrak{P}}^{*}E)$

($(\mathfrak{P}, K^{ab}/K)$: p-power Frobenius)

Pf. $Ell(R) = \{E_1, \dots, E_n\}$ with $E_i/\mathbb{Q}$
Take some finite Galois L/K

Let $\mathfrak{b}$ be an ideal with $\overline{\mathfrak{b}} = \overline{\mathfrak{p}}^{-1}$ in $Cl(R)$, $\mathfrak{P} \nmid \mathfrak{b}$
$p = \beta R$

$$\begin{array}{ccccccc}
\mathbb{C}/\Lambda & \xrightarrow[z \mapsto z]{} & \mathbb{C}/\mathfrak{p}^{-1}\Lambda & \xrightarrow[z \mapsto z]{} & \mathbb{C}/\mathfrak{b}^{-1}\mathfrak{p}^{-1}\Lambda = \mathbb{C}/\beta\Lambda & \xrightarrow[z \mapsto \beta^{-1}z]{} & \mathbb{C}/\Lambda \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow{\phi} & \overline{\mathfrak{p}}^{*}E & \xrightarrow{\psi} & \overline{\mathfrak{p}\mathfrak{b}}^{*}\Lambda & \xrightarrow{\qquad\lambda\qquad} & E
\end{array}$$

$\lambda \circ \psi \circ \phi$ is inseparable

$(\lambda \circ \psi \circ \phi)^{*}\omega_E = (\lambda \circ \psi \circ \phi)^{*}dz = d((\lambda \circ \psi \circ \phi)(z)) = d(\beta z) =$

$\beta\, dz = \beta\, \omega_E$

Reduce mod $\mathfrak{P}$.

$$(\tilde{\lambda}\circ\tilde{\tau}\circ\tilde{\phi})^*\tilde{\omega}_E = \tilde{\beta}\,\tilde{\omega}_{\tilde{E}} \bmod \mathfrak{P}$$

$$= \tilde{0} \bmod \mathfrak{P}$$

So $\tilde{\lambda}\circ\tilde{\tau}\circ\tilde{\phi}$ is inseparable

$\mathfrak{p}\mathfrak{b} = \beta R$, so $\mathfrak{P} \mid \mathfrak{p} \mid \beta$

$\deg(\tilde{\lambda}) = \deg(\lambda) = 1$ because isomorphism

$\deg(\tilde{\tau}) = \deg(\tau) = N_{K/\mathbb{Q}}(\mathfrak{b})$ ← prime to $\mathfrak{p}$

$\deg(\tilde{\phi}) = \deg(\phi) = N_{K/\mathbb{Q}}(\mathfrak{p}) = p$

$\tilde{\phi}$ is inseparable and has deg p

$$\tilde{E} \xrightarrow{\tilde{\phi}} \widetilde{\bar{\mathfrak{p}}^* E} \bmod \mathfrak{P}$$

Frob_p ↘ $\tilde{E}^{(p)}$ ↗ $\deg(l)$ (hence isomorphism)

$$j(\widetilde{\bar{\mathfrak{p}}^* E}) = j(\tilde{E}^{(p)}) = j(\tilde{E})^p \bmod \mathfrak{P}$$

$$j(E)^p \equiv j(E)^{(\mathfrak{p},L/K)} \bmod \mathfrak{P} \text{ by def of Artin map}$$

$\uparrow \sigma_{\mathfrak{p}}$ for convenience

$j(E)^p \equiv j(E)^{(p,\, 4/(c))} \equiv j(E^{\sigma_p}) \pmod{\mathfrak{P}}$

$j(\bar{\mathfrak{p}} * E) \equiv j(E^{\sigma_p}) \pmod{\mathfrak{p}}$

this is one of $j(E_1), \ldots, j(E_n)$

$j(\bar{\mathfrak{p}} * E) = j(E^{\sigma_p})$ since $\mathfrak{p}$ was assumed to be prime of good reduction

Lecture 36 (2011-05-06)

$E \in Ell(R)$ $R = \mathcal{O}_K$ Last time 2011-5-6 (36)

Thm. a) $H = K(j(E))$

b) $j(E)^{(a, H/K)} = j(\bar{a} * E)$

Goal: Generate K^{ab}

Analogy: $\mathbb{Q}^{ab} = \bigcup_{n=1}^{\infty} \mathbb{Q}(\zeta_n)$ $\zeta_n^{((a), \mathbb{Q}^{ab}/\mathbb{Q})} = \zeta_n^{|a|}$ for $(a, n) = 1$

Complication: adding in torsion points is not guaranteed to create abelian extensions

$P \in E_{tors}$ $H(P)$ / H / K not abelian

fix this by only adding x coordinate

cause: automorphisms of $\bar{E}$

Fix $E: y^2 = x^3 + Ax + B$ $A, B \in H = K(j(E))$

Def. $P \in E(\bar{K})$, $h(P)$ (not height) $= \begin{cases} x(P) & \text{if } AB \neq 0 \\ x(P)^3 & \text{if } A = 0 \\ x(P)^2 & \text{if } B = 0 \end{cases}$

Thm. Let $\mathfrak{d} \subset R$ be an ideal. Then $K(j(\bar{E}), h(\bar{E}[\mathfrak{d}]))$

is the ray class field of K of conductor $\mathfrak{d}$ (or maybe dividing $\mathfrak{d}$)

abelian
- $H(E[\vec{a}])$ — if $AB \neq 0$, deg $= 2^r$
- $H(h(E[\vec{a}]))$
- $K(j(E)) = H$ — abelian

looks like a generalized dihedral group

Lemma For all but finitely many degree 1 primes $\mathfrak{p}$ (i.e. principal, since $Cl(K) \xrightarrow{\sim} Gal(H/K)$, $(\cdot, H/K)$) in K satisfying $(\mathfrak{p}, H/K) = 1$, there is a unique $\pi_\mathfrak{p} \in R$ such that $\mathfrak{p} = \pi_\mathfrak{p} R$

$$\begin{array}{ccc} E & \xrightarrow{[\pi_\mathfrak{p}]} & E \\ \downarrow & & \downarrow \\ \tilde{E} & \xrightarrow{\text{Frob}_\mathfrak{p}} & \tilde{E} \end{array}$$ commutes.

Pf. See book

Pf of Thm Let $L = K(j(E), h(E[\vec{a}]))$

$H = K(j(E))$

K

To show: For all but finitely many degree 1 primes $\mathfrak{p}$,

$$(\mathfrak{p}, L/K) = 1 \iff \underbrace{\mathfrak{p} = \mu R,\ \mu \equiv 1 \bmod \mathfrak{c}}_{\mathfrak{p} \in P_1(\mathfrak{c})}$$

First: $\mathfrak{p} \in P_1(\mathfrak{c}) \Rightarrow (\mathfrak{p}, L/K) = 1$

We're given $\mathfrak{p} = \mu R$, $\mu \equiv 1 \bmod \mathfrak{c}$

Lemma gives $\pi \in R$, $\mathfrak{p} = \pi R$ such that

$$\begin{array}{ccc} E & \xrightarrow{[\pi]} & E \\ \downarrow & & \downarrow \\ \tilde{E} & \xrightarrow{\mathrm{Frob}_{\mathfrak{p}}} & \tilde{E} \end{array}$$

So $\pi = \{\mu$ for $\{ \in R^\times$. To Show: $(\mathfrak{p}, L/K)$ acts trivially on $\psi(E[\mathfrak{c}])$

Let $T \in E[\mathfrak{c}]$

$$\widetilde{T^{(\mathfrak{p}, L/K)}} = \tilde{T}^{\mathrm{Frob}_{\mathfrak{p}}} = \widetilde{[\pi](T)}$$

$\in E_{\text{ns}}$

$\in E_{\text{ns}}$

Erased

Now for other direction:

$1 = (\mathfrak{p}, L/K) \Rightarrow 1 = (\mathfrak{p}, L/K)|_H = (\mathfrak{p}, H/K)$

so $\mathfrak{p}$ is principal. The lemma says

$\mathfrak{p} = \pi R$, $\pi : E \to E$

$\downarrow \qquad \downarrow$

$\widehat{E} \xrightarrow{\text{Frob}_{\mathfrak{p}}} \widetilde{E}$

Choose some $\sigma \in \mathrm{Gal}(\overline{K}/K)$ with

$\sigma|_{K^{ab}} = (\mathfrak{p}, K^{ab}/K)$

$\sigma|_L = (\mathfrak{p}, L/K) = 1$

$\sigma|_H = 1$

Let $T \in E[\mathfrak{a}]$. Then $\widetilde{h([\pi]T)} =$

$h(\widetilde{[\pi]}\,\widetilde{T}) = h(\widetilde{T}^{\text{Frob}_{\mathfrak{p}}}) = h(\widetilde{T}^{(\mathfrak{p}, L/K)})$

by assumption $(\mathfrak{p}, L/K) = 1$, so this is $h(\widetilde{T})$

so $\widetilde{[\pi]}T = \widetilde{[\xi_T]}T$ for some $\xi_T \in R^\times$

$\widetilde{[\pi - \zeta_T]T} = \widehat{0} \Rightarrow [\pi - \zeta_T]T = 0$

$E[d]$ is a cyclic R/d-module, so because $[\pi - \zeta_T]$ kills $E[d]$, we have $\pi - \zeta \in d$, so $\pi \equiv \zeta \bmod d$, so

$$p = \pi R = \zeta R$$

Lecture 37 (2011-05-09)

$x \in \mathbb{A}_K^*$, $\alpha \in K^*$ 2011-5-9 (37)

want to define $x\alpha \in K^*$ (can't actually do this)

Lemma. $\mathfrak{a}$ a fractional ideal, can look at quotient group $K/\mathfrak{a}$

$$K/\mathfrak{a} \cong \bigoplus_{\mathfrak{p}} K_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$$

(simplest case: $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$)

Recall that we defined $(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x_{\mathfrak{p}})}$,

ideal corresponding to the adele x

Define
$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\cdot x} & K/(x)\mathfrak{a} \\ \| & & \| \\ \bigoplus K_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}} & \longrightarrow & \bigoplus K_{\mathfrak{p}}/((x)\mathfrak{a})_{\mathfrak{p}} \end{array}$$

$$(\alpha_{\mathfrak{p}} + \mathfrak{a}_{\mathfrak{p}}) \longmapsto (x_{\mathfrak{p}}\alpha_{\mathfrak{p}} + x\mathfrak{a}_{\mathfrak{p}})$$

Recall $[\,,K] : \mathbb{A}_K^* \to \mathrm{Gal}(K^{ab}/K)$

surjective; continuous; K^* is in the kernel

Main Theorem of Complex Multiplication (MTCM)

$K/\mathbb{Q}$ imaginary quadratic field, $R = \mathcal{O}_K$

$E \in \mathrm{Ell}(R)$. For an arbitrary $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$

choose $s \in \mathbb{A}_K^*$ such that $\sigma|_{K^{ab}} = [s, K]$.

Fix $f: \mathbb{C}/\mathfrak{a} \xrightarrow{\sim} E(\mathbb{C})$ with $\mathfrak{a} \subset K$.

Then $\exists f': \mathbb{C}/(s)^{-1}\mathfrak{a} \xrightarrow{\sim} E^{\sigma}(\mathbb{C})$ such

that this commutes

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\cdot s^{-1}} & K/s^{-1}\mathfrak{a} \\ f \downarrow & & \downarrow f' \\ E(\mathbb{C}) & \xrightarrow[\sigma]{} & E^{\sigma}(\mathbb{C}) \end{array}$$

Note: 1) $f(K/\mathfrak{a}) = E(\mathbb{C})_{tors}$

2) σ is not a $\mathbb{C}$-continuous map

3) $\underbrace{f(t)^{\sigma}}_{\text{Galois action}} = \underbrace{f'(s^{-1} \cdot t)}_{\text{analytic action}}$

Want to study how much $\bar{\mathbb{Q}}$ moves torsion points around

E/L, not necessarily with CM, consider $E[m]$

$\rho_m : G_{\bar{L}/L} \longrightarrow Aut(E[m])$

$\cong$ choose a basis

$GL_2(\mathbb{Z}/m\mathbb{Z})$

Thm (Serre) There is a constant $C_{E/L}$ such that $Aut(E[m]) / \rho(G_{\bar{L}/L}) \leq C_{E/L}$

(just cosets, since not necessarily normal)

actually only true if E does not have CM

(Serre curve: one with $C_{E/L} = 1$)

Conjecture: fix L. Then $\sup_{E/L} (C_{E/L}) < \infty$.

If E has CM, then (assuming $K \subset L$)

$G_{\bar{L}/L}$ commutes with $End(E)$

So $\rho_M(G_{L/L}) \subset Aut_R(E[M]) \cong (R/MR)^\times$

($E[M]$ is a free rank 1 R/MR-module)

Thm With E has CM, $\exists\, C_{E/L}$ such that

$$\#\frac{(R/MR)^\times}{\rho_M(G_{L/L})} \leq C_{E/L} \quad \text{for all } M \geq 1$$

Pf. **Case 1** Assume $L = K(j(E)) = H$

Choose some $a \in Aut_R(E[M]) = (R/MR)^\times$

Choose an $\alpha \in R$ with $\alpha \equiv a \bmod MR$

Choose $s \in \mathbb{A}_K^*$ with $(s) = \alpha^{-1}R$

s acts on $E[M]$ via $[\alpha]$

$s = [s, K]$ but only defined up to unit, i.e. up to some automorphism of E

$$G_{L/L} \xrightarrow{\rho_M} (R/MR)^\times \longrightarrow (R/MR)^\times/R^\times$$

(surjective)

needed $L = H$ so that

$E(\mathbb{C}) \longrightarrow E^\sigma(\mathbb{C})$ $= E$ in good form

For arbitrary L/K

$$[(K/\mathfrak{m}_K)^\times : \rho_{\mathfrak{m}}(G_{L/L})]$$

$$= [(K/\mathfrak{m}_K)^\times : \rho_{\mathfrak{m}}(G_{\bar{H}/H})][\rho_{\mathfrak{m}}(G_{\bar{H}/H}) : \rho_{\mathfrak{m}}(G_{L/L})]$$

$$\leq \#R^\times \cdot [G_{\bar{H}/H} : G_{L/L}]$$

$$= \#R^\times \cdot [L:H]$$