Math 252 - Algebra 2

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Introduction

Math 252 is one of the courses offered for mathematics graduate students at Brown University. It is the second of two courses in the year-long algebra sequence. I took these notes while auditing the course as a freshman.

The notes are handwritten because this was before I started live-TeXing. I may eventually get around to typing these notes properly.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to `chonoles@math.uchicago.edu`.

Lecture 1 (2009-01-27)

1/27/09

A <u>semi-local</u> if has finitely many maximal ideals

$\mathbb{Z} \subset \mathbb{Q}$, $\mathbb{Z}_p = \{a/b, p \nmid b\}$ local

$\mathbb{Z}_{(p_1 p_2 \cdots p_n)} = \{a/b, p_i \nmid b\}$ semilocal

PID: every ideal is principal

$\mathbb{Z}$, $k[T]$, $\mathbb{Z}[i]$

$k[T_1, \ldots, T_n]$ not PID if $n > 1$ $(J = (T_1, \cdots, T_n)$ not principal$)$

UFD: $x = u p_1^{n_1} \cdots p_u^{n_u}$, p_i distinct irred. elements

A UFD $\Rightarrow$ $A[T]$ UFD

PID $\Rightarrow$ every prime ideal $\neq (0)$ is maximal

Pf. $P = (a)$, $a \neq 0$, and suppose $P \subseteq M \subseteq A$

Let $M = (b)$, so $a = bc$, so $bc \in P$, so $b \in P$ or $c \in P$

$b \in P \Rightarrow M \subseteq P \Rightarrow M = P$

$c \in P = c = da$, $a = bda$, $a \neq 0 \Rightarrow 1 = ba$

$M = (b) = A$

$x \in A$ is nilpotent if $\exists n \in \mathbb{N}$ with $x^n = 0$

$N = \{x \in A : x \text{ nilpotent}\}$

$N =$ nilradical(A)

Prop. N is an ideal

multiplication is easy

addition: $x^r = 0$, $y^s = 0$, $(x+y)^{r+s} = 0$

Prop. $N = \bigcap P$, P prime in A

For any P prime, $x \in N \Rightarrow x^n = 0 \in P \Rightarrow x \in P$, so $N \subseteq P$ for all P, so $N \subseteq \bigcap P$

Other direction: f not nilpotent. Let Σ = set of ideals I with $f^n \notin I$ $\forall n$. $\Sigma \neq \emptyset$ since $(0) \in \Sigma$, so chain of ideals I_α in Σ $\bigcup I_\alpha \in \Sigma$. Zorn's lemma, so there exists a maximal P in Σ

Claim: P is prime.

Suppose $ab \in P$, $a \notin P$, $b \notin P$. $P + (a) > P$, $P + (b) > P$. $f^n \in P + (a)$, $f^m \in P + ($

$f^{n+m} \in P + (a)(b) > P + (ab) = P$, contradiction

Def. I ideal of A. Radical of I = $r(I) = \{x \in A : x^n \in I\}$

$r(I)$ is an ideal.

Prop. $r(I) = \bigcap_{P \supseteq I} P$ prime

Pf. Look at A/I. Nilradical$(A/I) = \bigcap Q$, Q prime in A/I

$\pi : A \to A/I$ $\quad \pi^{-1}(\text{nilradical}(A/I)) = r(I)$, $\pi^{-1}(\bigcap Q) = \bigcap P$

Def. Jacobson radical$(A) = \bigcap M$, M maximal in A.

If $A = \mathbb{Z}, k[T], k[T_1, \ldots, T_n]$, Jacobson radical = radical = (0)

If A local domain, only one max ideal M,

nil radical = (0), Jacobson rad = M

Operations on ideals

$\{I_j\}$ family of ideals

$\bigcap_j I_j$, $\sum_j I_j$ = intersection of all ideals containing all the I_j's

Set of finite sums $\sum \alpha_i$, α_i is in some I_j

$\prod_{j=1}^{n} I_j$ = ideal generated by $\alpha_1 \cdots \alpha_n$, $\alpha_j \in I_j$, i.e., set of finite sums of $\sum \alpha_1 \cdots \alpha_n$

Ex. $\mathbb{Z}$, or any PID

$(m)(n) = (mn)$

$(m) \cap (n) = (\operatorname{lcm}(m,n))$

$(m) + (n) = (\gcd(m,n))$

Def. I, J coprime if $I + J = A = (1) \iff \exists x \in I, y \in J, x + y = 1$

$IJ \subseteq I \cap J$, but if I, J coprime, $IJ = I \cap J$

Pf. $1 = a + b$ for $a \in I$, $b \in J$. $x \in I \cap J$. $x = x \cdot 1 = x(a+b) =$

$\underbrace{xa}_{\in IJ} + \underbrace{xb}_{\in IJ} \in IJ$

A_i ring for $i = 1, \ldots, n$

$\prod_{i=1}^{n} A_i$ a ring

$\phi : A \to \prod_{i=1}^{n} A/I_i$ for some ideals I_i

$\operatorname{Ker}(\phi) = \bigcap I_i$

1/27/09

Thm. I_i, I_j coprime for $i \neq j$ Iff ϕ is surjective

WLOG, let $i = 1$.

If ϕ is surjective, $\exists\, x : \phi(x) = (1, 0, \ldots, 0)$, i.e. $x \mapsto 1 \in A/I_1$

so $x - 1 \in I_1$, $x \in I_j$ for $j \geq 2$

$x - 1 = -a \in I_1$

$1 = a + x$

$a \in I_1$, $x \in I_j$

Other direction: Suppose I_i, I_j coprime; suffices to show $(1, 0, 0, \ldots, 0)$ in image of ϕ (since the order of the coordinates isn't special, so we can get any $(0, 0, \ldots, 1, 0, 0)$, and these generate the whole thing)

$x_j \in I_1$, $y_j \in I_j$ $\quad 1 = x_j + y_j$

~~$y_j z =$~~

If I_i, I_j coprime for $i \neq j$, $\prod_{j=1}^{n} I_j = \bigcap_{j=1}^{n} I_j$

Pf. Know this for 1 (duh), 2 (we did this before)

Let $J = \prod_{j=1}^{n-1} I_j = \bigcap_{j=1}^{n-1} I_j$ (by induction)

$I_j + I_n = (1)$ $\quad x_j + y_j = 1$

$x_j \in I_j$, $y_j \in I_n$

$\prod x_j = \prod (1 - y_j) \equiv 1 \bmod I_n$

$\Rightarrow J + I_n = (1)$

Thus $J I_n = \prod_{j=1}^{n} I_j = \bigcap_{j=1}^{n} I_j = J \cap I_n$

Suppose $P_1, \dots, P_n$ prime ideals, $\underset{\text{ideal}}{I} \subseteq \bigcup_{i=1}^{n} P_i$, then $\exists i : I \subseteq P_i$

Pf. $I \not\subseteq P_i \Rightarrow I \not\subseteq \bigcup P_i$, use induction on n

True for $n-1$, $\exists x_j : x_j \in I$, $x_j \notin P_i$ for $i \neq j$

If $x_j \notin P_j$, then done. If $x_j \in P_j$, consider

$$X = \sum_{k=1}^{n} x_1 \cdots x_{k-1} x_{k+1} \cdots x_n \quad , \quad x \in I,$$

but $x \notin P_j$ for all j since $x_1 \cdots x_{j-1} x_{j+1} \cdots x_n \notin P_j$ but all other terms are

H/W. 1.12, 1.13, 1.18 (don't hand in) $\longleftarrow$ in the text

page 10 problem 1, page 12 problem 15

due next week

Lecture 2 (2009-01-29)

1/29/09

$r(I+J) = r(r(I)+r(J))$

Recall, $r(I) = \{x : x^n \in I \text{ for some } n\}$. Obviously $I \subseteq r(I)$ and $J \subseteq r(J)$, so $I+J \subseteq r(I)+r(J)$, so $r(I+J) \subseteq r(r(I)+r(J))$.

Let $x \in r(r(I)+r(J))$. Then

$x^n \in r(I)+r(J)$ so

$x^n = y+z$ for $y \in r(I)$, $z \in r(J)$

so $y^m \in I$, $z^s \in J$, so $(y+z)^{m+s} \in I+J$

so $x^{n(m+s)} \in I+J$, so $x \in r(I+J)$.

Cor: $I+J=(1) \iff r(I)+r(J)=(1)$

Pf. $I+J=(1) \iff r(I+J)=(1) \iff r(r(I)+r(J))=(1) \iff r(I)+r(J)=(1)$

If $f: A \to B$ is a ring homomorphism, I an ideal of A, then

I^e = extended ideal = $f(I)B$ = smallest ideal in B containing $f(B)$

Similarly, J an ideal of B, J^c = contracted ideal = $f^{-1}(J)$ is an ideal of A

$x \in f^{-1}(J)$ $a \in A$

$f(ax) = f(a)f(x) \in J$

P prime ideal of $B \Rightarrow f^{-1}(P) = P^c$ is prime

so A a ring, $\mathrm{Spec}(A) = \{\text{prime ideals of } A\}$, $f: A \to B$ we get

$$f^{-1} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

An A-module M is an abelian group M under $+$ with

$a_1(a_2 m) = (a_1 a_2)m$

$1m = m$

$a(m_1+m_2) = am_1 + am_2$

$(a_1+a_2)m = a_1 m + a_2 m$

A is an A-module.

A is a field, M is a vector space

$A = \mathbb{Z}$, M is ab. gp.

If I ideal of A, I is A module

$A = F[T]$, V is a vector space with an F-endomorphism

Category : collection of objects (class)

for each $X, Y \in ob(C)$, a set $Hom(X, Y)$ with

$\exists\ id_X \in Hom(X, Y)$

$\exists \circ : Hom(X,Y) \times Hom(Y,Z) \xrightarrow{\circ} Hom(X,Z)$

with $Hom(X,X) \times Hom(X,Y) \to Hom(X,Y)$

$id_X, \quad f \longmapsto f$

$Hom(X,Y) \times Hom(Y,Y) \to Hom(X,Y)$

$f, \quad id_Y \longmapsto f$

$f : X \to Y$ is an isomorphism if $\exists\ g : Y \to X$: $f \circ g = id_Y$ and $g \circ f = id_X$

In Ab, vector spaces, A-mod, f is an isomorphism $\Leftrightarrow$ f is a bijection

not in Top : $[0,1) \to S_1$

Category of categories

T is a functor from $A \to B$

$T(Ob(A)) = Ob(B)$

$f : A_1 \to A_2$ $\quad T(f) = T(A_1) \to T(A_2)$ covariant

$T(f) : T(A_2) \longrightarrow T(A_1)$ contravariant

A = top spaces, B = abelian groups

$S \in Ob(A)$

$T(S) = H_i(A, \mathbb{Z})$ homology covariant

$C(S) = H^i(A, \mathbb{Z})$ cohomology contravariant

A = pointed top spaces, B = groups

$V(X, x_0) = \pi_1(X, X_0)$

base point $\longrightarrow$

M, N are A-mods

$Hom_A(M, N) = \{$A-mod homs from M to N$\}$

$Hom_A(M, N)$ is an A-mod

$(f+g)(x) = f(x) + g(x)$

$(af)(x) = a\, f(x)$

only works if A is comm.

If we fix an A-module M, $N \mapsto \mathrm{Hom}_A(M,N)$ is a covariant functor from A-mod to A-mod

1/29/09

$$\begin{tikzcd}
N_1 \arrow[r, "f"] & N_2 \\
M \arrow[u, "g"] \arrow[ur, "f \circ g"']
\end{tikzcd}$$

$g \mapsto f \circ g$

$\mathrm{Hom}_A(M,N_1) \to \mathrm{Hom}_A(M,N_2)$

If we fix N, $M \mapsto \mathrm{Hom}_A(M,N)$ is contravariant

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[dr] & M_2 \arrow[d, "g"] \\
 & N
\end{tikzcd}$$

$g \mapsto g \circ f$

$$\begin{tikzcd}
\mathrm{Hom}_A(M_2, N) \arrow[d] \\
\mathrm{Hom}_A(M_1, N)
\end{tikzcd}$$

N is a submodule of M if its an A-mod under operations of M

$A = F$ subspaces

$A = \mathbb{Z}$ subgroups

quotient modules: $x \sim y$ if $x - y \in N$ a submodule

M/N = set of equivalence classes

$f: M \to N$

$\mathrm{Ker}(f) = \{f(x) = 0\}$

$\mathrm{Im}(f)$ $\qquad$ $\mathrm{Coker}(f) = N/\mathrm{Im}(f)$

$M/$

Lecture 3 (2009-02-03)

2/3/09

F is a free A-mod if $F \simeq \coprod_{i \in I} A$

$\mathrm{Hom}_A(F_I, P) = \mathrm{Hom}_A(\coprod A, P) = \prod_A \mathrm{Hom}_A(A, P) = \prod_A P$

$\mathrm{Hom}_A(P, M \oplus_A N) = \mathrm{Hom}_A(P, M) \times \mathrm{Hom}_A(P, N)$

$\Pi : P \to M \oplus N$

$p_1 : M \oplus N \to M$, $p_2 : M \oplus N \to N$

$\Pi \longmapsto (p_1 \circ \Pi, p_2 \circ \Pi)$

$P \xrightarrow{f} M$, $P \xrightarrow{g} N$

$\Pi(p) = (f(p), g(p))$

ISO THMS

$L \subseteq M \subseteq N$

$N/M \simeq {}^{N/L}\big/_{M/L}$

Tip: to show $M/N \simeq P$,

Give $f : M \to P$, show it is surjective, and $\mathrm{Ker}(f) = N$

$N \twoheadrightarrow N/L \twoheadrightarrow {}^{N/L}\big/_{M/L}$

$\dfrac{M_1 + M_2}{M_1} \simeq \dfrac{M_2}{M_1 \cap M_2}$

$M_2 \longrightarrow \dfrac{M_1 + M_2}{M_1}$, with $m_2 \mapsto [M_2]$

Kernel $= M_2 \cap M_1$

$M_i \subseteq M$, submodules of an A-mod M

$\bigcap M_i$ is submod

$\sum M_i$ is submod

Can't do product of M_i, but can do

$IM = \sum_{i=1}^{n} a_i m_i$ for ideal I of A, $a_i \in I$, $m_i \in M$

M is finitely generated iff $\exists$ surjective map $f : A^n \twoheadrightarrow M$.

($\Rightarrow$) If M is generated by $m_1 \ldots m_n$, the map $A^n \to M$ by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i m_i$

Clearly surjective

($\Leftarrow$) $f : A^n \to M$ surjective. Let

$m_i = f(0, \ldots 0, 1, 0 \ldots)$

m_i generate

classical adjoint of a matrix over A $= T^*$

T is an $n \times n$ matrix over A, $T = (a_{ij})$

classical adjoint of $T = U$ where $U_{ij} = (-1)^{i+j} \det T(j/i)$

$T(j/i) =$ remove j^{th} row, i^{th} column

Thm. $TT^* = T^*T = \det(T) I_n$

Thm. Let M be a f.g. A-module. Let I be an ideal of A, let $\phi : M \to M \in \mathrm{Hom}_A(M,M) = \mathrm{End}_A(M)$, and assume $\phi(M) \subseteq IM$.

$\Rightarrow \phi$ satisfies an equation of the form $\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_1\phi + a_0 = 0$,

for $a_i \in \underline{I}$. This is an equation in $\mathrm{End}_A(M)$, or can be viewed as for all $x \in M$, $\phi^n(x) + a_{n-1}\phi^{n-1}(x) + a_1\phi(x)$ $a_0x =$ $\in M$

Pf. Let $x_1, \ldots, x_n$ be a set of generators of M.

$\phi(x_i) = \sum_{j=1}^{n} a_{ij} x_j \quad a_{ij} \in I$, so $\sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij}) x_j = 0$

~~Let $A_{ij} = \delta_{ij}\phi - a_{ij}$~~

where a_{ij} is now considered as multiplication by $a_{ij} \in \mathrm{End}_A(M)$

The commutative ring we are considering is the subring of $\mathrm{End}_A(M)$ generated by "multiplications by A" and ϕ, which is all the terms $\sum a_i \phi^i \subset \mathrm{End}_A(M)$. This ring is R.

Let $A_{ij} = \delta_{ij}\phi - a_{ij} \in R$.

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ A_{n1} & & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_n \end{pmatrix} = 0$$

$$AX = 0$$

Thus $A^*AX = 0$, But $A^*A = \det(A) I_n$, so $\det(A)X = 0$,

where $\det(A)X = \begin{pmatrix} (\det(A))x_1 \\ \vdots \\ \det(A)x_n \end{pmatrix} = 0 \Rightarrow \det(A)x_i = 0 \ \forall i$,

so $\det(A) = 0 \in \mathrm{End}_A(M)$

$$A = \begin{pmatrix} \phi - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & & & \\ \vdots & & & \\ -a_{n1} & & \cdot & \phi - a_{nn} \end{pmatrix}$$

$\det(A) = \phi^n + a_{n+1}\phi^{n-1} + \cdots + a_1\phi + a_0 = 0 \quad a_i \in I$

Corollary. M a f.g. A-mod, I an ideal of A, $IM = M$

$\exists x \equiv 1 \ (I) : xM = 0$

Pf. Take $\phi = id$, $\phi^n + a_{n-1}\phi^{n-1} + \cdots a_0 = 0$, $a_i \in I$

Let $x = 1 + a_{n-1} + \cdots + a_1 + a_0 \quad xM = 0$

Nakayama's Lemma

Prop. M f.g. A-mod, I ideal $\subseteq J(A) = \bigcap_{m \text{ max}} m$

$IM = M \Rightarrow M = 0$

Pf1. $\exists x \equiv 1 \ (I) : xM = 0 \Rightarrow x$ unit (x not in any m)

$x^{-1}xM = 0 \Rightarrow M = 0$

Pf 2. Take minimal set of generators for M, $u_1, \ldots, u_n$

$u_1 = a_1u_1 + \cdots + a_nu_n \quad a_i \in I$

$\underset{\uparrow \text{ unit}}{(1 - a_1)}u_1 = a_2u_2 + \cdots a_nu_n$

So $u_1 = (1-a_1)^{-1}a_2u_2 + \cdot (1-a_1)^{-1}a_nu_n$

$\Rightarrow u_2, \ldots, u_n$ generate $M \Rightarrow M = 0$

contradiction

Lecture 4 (2009-02-05)

2/5/09

A a local ring, M an f.g. A-mod, m the max. ideal of A,

$$M/mM = 0 \Rightarrow M = 0$$

Cor. A local, M an f.g. A-mod, $x_1, \dots, x_n \in M$ such that $\bar{x}_1, \dots, \bar{x}_n$ generate M/mM as an A/m module. Then $x_1, \dots, x_n$ generate M as an A-module.

Pf. Let $N < x_1, \dots, x_n > \subseteq M$. Want $N = M$. We know $N + mM = M$ since $\bar{N} = \bar{M}$.

$$\frac{M/N}{m(M/N)} \cong \frac{M}{(N+mM)}$$

$$\longleftarrow M$$

M surjects onto M/N which surjects onto $M/N/m(M/N)$

Kernel of $M \to M/N$ is N; if $x \in M$ not in N, but in mM, gets killed going to $m(M/N)$. By previous, $M/N = 0 \Rightarrow M = N$

Exact sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \quad M_i \ A\text{-mod's}$$

$$f_i : M_i \to M_{i+1} \in \mathrm{Hom}_A(M_i, M_{i+1})$$

Exact if $\mathrm{Ker}(f_{i+1}) = \mathrm{Im}(f_i)$

a complex if $\mathrm{Im}(f_i) \subseteq \mathrm{Ker}(f_{i+1})$

$$\Updownarrow$$

$$f_{i+1} \circ f_i = 0$$

The functor $M \mapsto \mathrm{Hom}_A(M, N)$ is left exact, which is to say

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0 \text{ exact} \Rightarrow$$

$$0 \to \mathrm{Hom}_A(M_3, N) \xrightarrow{\phi} \mathrm{Hom}_A(M_2, N) \xrightarrow{\psi} \mathrm{Hom}_A(M_1, N) \quad \text{exact}$$

1) ϕ injective

$$\begin{tikzcd} g : M_3 \arrow[r] & N \\ M_2 \arrow[u, "f_2"] & \end{tikzcd}$$

$x \in M_3$

$x = f_2(y)$ since f_2 surjective

$g(x) = g(f_2(y)) = 0$

$\phi(g) = g \circ f_2$

$g \circ f_2 = 0 \Rightarrow g = 0$

2) $\mathrm{Im}(\phi) = \mathrm{Ker}(\psi)$

$\mathrm{Im}(\phi) \subseteq \mathrm{Ker}(\psi)$ since

$$\begin{tikzcd} M_1 \arrow[d, "f_1"] \arrow[dr] & \\ h : M_2 \arrow[r] \arrow[d, "f_2"] & N \\ M_3 \arrow[ur, "g"'] & \end{tikzcd}$$

$h = g \circ f_2$

$\psi(h) = h \circ f_1 = g \circ f_2 \circ f_1 = g \circ 0 = 0$

$\mathrm{Ker}(\psi) \subseteq \mathrm{Im}(\phi)$

$$\begin{tikzcd}
M_1 \arrow[d, "f_1"'] \arrow[dr] & \\
M_2 \arrow[r] \arrow[d] & N \\
M_3 \arrow[d] & \\
0 &
\end{tikzcd}$$

$\psi(h) = h \circ f_1 = 0$

$h(\mathrm{Im}(f_1)) = 0$

gives

$$\begin{tikzcd}
M_3 = M_2/\mathrm{Im}(f_1) \arrow[d] \\
N
\end{tikzcd}$$

call $0 \to \mathrm{Hom}_A(M_3, N) \xrightarrow{\phi} \mathrm{Hom}_A(M_2, N) \xrightarrow{\psi} \mathrm{Hom}_A(M_1, N)$ by $(*)_N$

call $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ by $(*)$

Claim. $(*)_N$ exact $\forall N \Rightarrow (*)$ exact

a). $N = M_3/f_2(M_2)$

$\mathrm{Hom}(M_3, M_3/f_2(M_2)) \to \mathrm{Hom}(M_2, M_3/f(M_2))$ is injective

$$\begin{tikzcd}
M_2 \arrow[r, "f_2"] \arrow[dr, "0"'] & M_3 \arrow[d, "\pi"] \\
& M_3/f_2(M_2)
\end{tikzcd}$$

$\pi \mapsto 0 \Rightarrow \pi = 0$

$\Rightarrow M_3/f_2(M_2) = 0 \Rightarrow f_2$ surjective

$\Rightarrow f_2(M_2) = M_3$

b). Now we want $\mathrm{Im}(f_1) \subseteq \mathrm{Ker}(f_2)$, i.e. $f_2 \circ f_1 = 0$.

Let $N = M_3$, $g = id \in \mathrm{Hom}_A(M_3, M_3)$

$\chi \circ \phi(g) = 0$ so

$g \circ f_2 \circ f_1 = 0$. but $g = id$, so $f_2 \circ f_1 = 0$

c) $N = M_2/\mathrm{Im}(f_1)$

$$\begin{tikzcd}
M_3 \arrow[dr, bend left, "h"] & \\
M_2 \arrow[u, "f_2"] \arrow[r, "g"'] & M_2/\mathrm{Im}(f_1) \\
M_1 \arrow[u, "f_1"] \arrow[ur] &
\end{tikzcd}$$

$g =$ projection

$\psi(g) = 0$

$=$

$g \circ f_1$

since $\psi(g) = 0$, $g = \phi(h)$.

$h : M_3 \to M_2/\mathrm{Im}(f_1)$

$g = h \circ f_2$

Thus $\mathrm{Ker}(f_2) \subseteq \mathrm{Ker}(g) = \mathrm{Im}(f_1)$

2/5/09

Can similarly prove $0 \to M_1 \to M_2 \to M_3$ exact

$$\Longleftrightarrow$$

$$0 \to \mathrm{Hom}_A(N, M_1) \to \mathrm{Hom}_A(N, M_2) \to \mathrm{Hom}_A(N, M_3) \text{ exact } \forall N$$

$\mathrm{Hom}_A(N, -)$ is covariant left exact (other was contravariant)

N is injective if $M \to \mathrm{Hom}_A(M, N)$ is exact

N is projective if $M \to \mathrm{Hom}_A(N, M)$ is exact

A covariant functor T is exact if

1) $\forall$ $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ exact $\Rightarrow$ $T(M_1) \to T(M_2) \to T(M_3)$ exact

2) $\dots M_{-1} \to M_0 \to M_1 \to M_2 \to \dots$ exact $\Longleftrightarrow$

$\dots T(M_{-1}) \to T(M_0) \to T(M_1) \to T(M_2) \to \dots$ exact

3) $0 \to M_1 \to M_2 \to M_3 \to 0$ exact $\Longleftrightarrow$ $0 \to T(M_1) \to T(M_2) \to T(M_3) \to 0$ exactly

These 3 def's are equivalent

If $M_{-1} \xrightarrow{f_{-1}} M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$ exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f_1) \arrow[r] & M_1 \arrow[r] \arrow[d] & \mathrm{Im}(f_1) \arrow[r] \arrow[dll, bend left] & 0 \\
0 \arrow[r] & \mathrm{Ker}(f_2) \arrow[r] & M_2 \arrow[r] \arrow[d] & \mathrm{Im}(f_2) \arrow[r] \arrow[dll, bend left] & 0 \\
0 \arrow[r] & \mathrm{Ker}(f_3) \arrow[r] & M_3 \arrow[r] & \mathrm{Im}(f_3) \arrow[r] & 0
\end{tikzcd}$$

but $\mathrm{Im}(f_1) = \mathrm{Ker}(f_2)$
$\mathrm{Im}(f_2) = \mathrm{Ker}(f_3)$

Same thing but with T's on everything

Given A a ring, M, N A-mods, we want an A-mod $M \otimes_A N$ together with an A-bilinear map $M \times N \xrightarrow{T} M \otimes_A N \xrightarrow{f} P$ such that if P is any A-module $\mathrm{Hom}_A(M \otimes_A N, P) \xrightarrow{\sim} \mathrm{Bilin}_A(M \times N, P)$ given by $f \mapsto f \circ T$ is a bijection

Construction : Take free A-module on $M \times N$, $A^{M \times N}$, has one generator for each pair (m,n)

Elements are finite sums $\sum_i a_i \langle m_i, n_i \rangle$

$M \times N \to A^{M \times N}$

$(m,n) \mapsto \langle m, n \rangle$

Let J = submodule of $M \times N$ generated by all

$$\langle m_1 + m_2, n \rangle - \langle m_1, n \rangle - \langle m_2, n \rangle$$

$\vdots$ other relations

Let $M \otimes_A N = A^{M \times N} / J$

$T((m,n)) = [\langle m, n \rangle]$

$M \times N \xrightarrow{B} P$, B bilinear

$U(M \otimes_A N) = U([\langle m n \rangle]) = B(m,n)$

R_1 = candidate for $M \otimes_A N$

by universality

$U \circ T_2 = T_1$

$V \circ T_1 = T_2$

$$\begin{tikzcd} M \times N \arrow[r, "T_1"] \arrow[dr, "T_2"'] & R_1 \arrow[d, shift left, "V"] \\ & R_2 \arrow[u, shift left, "U"] \end{tikzcd}$$

$V \circ U = id_{R_2}$

$V \circ U \circ T_2 = T_2$

$U \circ V = id_{R_1}$

Lecture 5 (2009-02-10)

2/10/09

A ring, M, N, P A-mods

Prop. $M \otimes_A N \simeq N \otimes_A M$ $\left(\begin{array}{l} M \times N \to N \otimes M \\ (m, n) \mapsto n \otimes m \end{array} \right.$ which induces a map $\begin{array}{l} M \otimes_A N \to N \otimes_A M \\ m \otimes n \mapsto n \otimes m \end{array}$ similarly in other direction, comp is identity, so iso$\Big)$

$M \otimes_A (N \otimes_A P) \simeq (M \otimes_A N) \otimes_A P \simeq M \otimes_A N \otimes_A P$

$M \otimes_A (N \oplus P) \simeq (M \otimes_A N) \oplus (M \otimes_A P)$

$M \otimes_A A \simeq M$

Similarly, $\begin{array}{l} M \times A \to M \\ (m, a) \mapsto am \end{array}$ $\begin{array}{l} M \to M \otimes A \\ m \mapsto m \otimes 1 \end{array}$

$\begin{array}{l} M \otimes A \to M \\ m \otimes a \mapsto am \end{array}$ $\begin{array}{l} M \to M \otimes A \to M \\ m \mapsto m \otimes 1 \mapsto m \end{array}$ ✓

$M \otimes A \to M \to M \otimes A$

$m \otimes a \mapsto am \mapsto am \otimes 1 = a(m \otimes 1) = m \otimes a$ ✓

<u>Change of rings</u>

$A \xrightarrow{f} B$ (B is an A-algebra)

B-modules $\overset{U}{\underset{\text{functor}}{\rightsquigarrow}}$ A-modules $U(M)$ is an A-mod via f

$a * m \overset{\text{def}}{=} f(a)m$

T: A-modules $\rightsquigarrow$ B-modules

$M \rightsquigarrow B \otimes_A M$

$b(b_1 \otimes m) = bb_1 \otimes m$

$$\begin{tikzcd} M_1 \arrow[d, "g"'] & \rightsquigarrow & B \otimes_A M_1 \arrow[d] \\ M_2 & & B \otimes_A M_2 \end{tikzcd}$$

$g(b \otimes m_1) = b \otimes g(m_1)$

$g(b_1 m_1) = b\, g(m_1)$

Prop. let M be A-mod, N be B-mod

$\mathrm{Hom}_B(B \otimes_A M, N) \simeq \mathrm{Hom}_A(M, U(N))$

i.e. $\mathrm{Hom}_B(T(M), N) \simeq \mathrm{Hom}_A(M, U(N))$ T, U in this situation called adjoint functors

T is left adjoint to U

U is right adjoint to T

Prop. N a f.g. B-mod, M a $\underset{\text{and}}{\text{f.g.}}$ A-mod, then N is a f.g. A-mod

M a f.g. A-mod, N a f.g. B-mod, then $B \otimes_A M$ is a f.g. B-mod

$N = \langle n_1, \ldots, n_r \rangle$

$B = \langle b_1, \ldots, b_s \rangle$

$\langle b_i n_j \rangle$

$A \xrightarrow{f} B$ 1) B is a f.g. A-module when $\exists\, b_1, \ldots, b_r \in B$:
$b \in B \Rightarrow b = \sum f(a_i) b_i$

2) B is an f.g A-algebra when $\exists\, b_1, \ldots, b_r \in B$: $b \in B$ is a polynomial in $b_1, \ldots, b_r$ with coeffs in $f(A)$

Example: $B = A[T]$ f.g. as an A-algebra, (by T), but not as A-module

Prop. M, N, P A-modules
$\mathrm{Hom}_A(M \otimes_A N, P) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ (canonically)

Let $f \in \mathrm{Hom}_A(M \otimes_A N, P)$. $f \longrightarrow g$ when $g(m)(n) = f(m \otimes n)$

Let $g \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,P))$. $g \longrightarrow f$ where $f(m \otimes n) = g(m)(n)$

$\tilde{f}(\langle m, n\rangle) = g(m)(n)$ (check it's bilinear)

Corollary. $- \otimes_A N$ is right exact, i.e.
if $M_1 \to M_2 \to M_3 \to 0$, is
$M_1 \otimes_A N \to M_2 \otimes_A N \to M_3 \otimes_A N \to 0$ exact?

Enough to know that, if we Hom it into a fixed P, that the r.s. exact, i.e.
suffices that $0 \to \mathrm{Hom}_A(M_3 \otimes_A N, P) \to \mathrm{Hom}_A(M_2 \otimes_A N, P) \to \mathrm{Hom}_A(M_1 \otimes N, P)$ exact $\forall P$,
but this is $0 \to \mathrm{Hom}_A(M_3, \mathrm{Hom}_A(N,P)) \to \mathrm{Hom}_A(M_2, \mathrm{Hom}_A(N,P)) \to \mathrm{Hom}_A(M_1, \mathrm{Hom}_A(N,P)))$
which we know is exact since $M_1 \to M_2 \to M_3 \to 0$ is exact.

Let $T(M) = M \otimes_A N$, $U(P) = \mathrm{Hom}_A(N, P)$; this says $\mathrm{Hom}_A(T(M), P) \simeq \mathrm{Hom}_A(M, U(P))$.

In fact, <u>any</u> left adjoint is right exact, and any right adjoint is left exact.

Exercise (not hand in). Suppose T a $\underset{\text{right-exact}}{\overset{\text{additive}}{\text{functor}}}$ A-mod $\to$ A-mod, and
T commutes with direct sums, i.e. $T(M \oplus N) = T(M) \oplus T(N)$.
Then $T(M) \simeq T(A) \otimes_A M$

$\otimes_A$ product not exact in general (it is if A is field)

$\otimes_A A$ exact (since it's the identity functor)

$\otimes_A A^n$ exact over K field M f.g. $M \simeq K^n$

2/10/09

Example of non-exact tensor product

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{exact}$$

$$\downarrow \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{obviously not exact}$$

Def. M is flat $/A$ $\Leftrightarrow$ $\otimes_A M$ is exact

Ex. If $A = \mathbb{Z}$, M flat $\Leftrightarrow$ M torsion-free

M f.g. $\Leftrightarrow$ free.

Suppose
$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B & & C \end{tikzcd}$$
, so B, C are A-algebras. $B \otimes_A C$ is an A-algebra.

$a \mapsto f(a) \otimes g(a)$ NO GOOD (Atiyah Macdonald gets this wrong)

$a \mapsto f(a) \otimes 1 = 1 \otimes g(a)$, $a(b \otimes c) = ab \otimes c = b \otimes ac$

$$ab \otimes c \;\; \| \;\; f(a)b \otimes c = b \otimes g(a)c$$

Prop. $B, C \rightsquigarrow B \otimes_A C$ is the direct sum in the category of rings

$$\begin{tikzcd} b \arrow[r, mapsto] & b \otimes 1 & \\ B \arrow[r] & B \otimes_A C \arrow[r] & D \text{ any } A\text{-alg} \\ C \arrow[ur] & 1 \otimes c & \\ c \arrow[ur, mapsto] & & \end{tikzcd}$$

$$\mathrm{Hom}_A(B \otimes_A C, D) = \mathrm{Hom}_A(B, D) \times \mathrm{Hom}_A(C, D) \qquad A\text{-algebras}$$

If $\phi: B \to D$, $\psi: C \to D$, define $(\phi \otimes \psi)(\langle b, c \rangle) = \phi(b) \otimes \psi(c)$

$\downarrow$ bilinearity

addivity is easy, check multiplicativity $\quad (\phi \otimes \psi)(b \otimes c) = \phi(b) \otimes \psi(c)$

$$(\phi \otimes \psi)((b_1 \otimes c_1)(b_2 \otimes c_2)) = (\phi \otimes \psi)(b_1 b_2 \otimes c_1 c_2) = \phi(b_1 b_2) \otimes \psi(c_1 c_2) =$$
$$\phi(b_1)\phi(b_2) \otimes \psi(c_1)\psi(c_2) = (\phi(b_1) \otimes \psi(c_1))(\phi(b_2) \otimes \psi(c_2))$$

Ex. $A[T] \otimes_A A[U] \simeq A[T, U]$

If $A \to B$, $A[T] \otimes_A B \simeq B[T]$

$$\begin{tikzcd} & L & \\ E \arrow[ur, hook] & & F \arrow[ul] \\ & k \arrow[ul, "\text{finite}"] \arrow[ur, "\text{finite}"'] & \end{tikzcd}$$

$E \otimes_k F \twoheadrightarrow EF$ composite

surjective

$$\mathbb{Q}(\sqrt{2}) \otimes \mathbb{Q}(\sqrt{3}) \xrightarrow{\sim} \mathbb{Q}(\sqrt{2}, \sqrt{3})$$

Direct limits.

I is a directed set if I is a poset and

$\forall a, b \in I, \exists c \in I : c \geq a, c \geq b$

$a \geq b, b \geq a \Rightarrow a = b$

$M_i, i \in I$ is directed family of A-modules if $\exists$ maps $\phi_{ji} : M_i \to M_j$ such that $\Rightarrow$

$$\begin{tikzcd}
M_i \arrow[r, "\phi_{ji}"] \arrow[dr, "\phi_{ki}"'] & M_j \arrow[d, "\phi_{kj}"] \\
 & M_k
\end{tikzcd}$$

Commutes $\phi_{ki} = \phi_{kj} \circ \phi_{ji}$

also written $\coprod$

Def. $\varinjlim\limits_{i\in I} M_i = \bigoplus\limits_{i\in I} M_i \Big/ \left(\langle m_i \rangle - \phi_{ji} \langle m_i \rangle \right)$

$\langle m_i \rangle$ denotes $(0, 0, \dots m_i, \dots 0)$

Every element in the direct limit $= \langle m_j \rangle$ some j, some $m_j \in M_j$

If $m_j \mapsto 0$ in $\varinjlim M_i$ then $\exists k \geq j : m_j \mapsto 0$ in M_k

Lecture 6 (2009-02-12)

2/12/09

2.20 (page 29), p.31 #1, 2, 3, 9, 18, 19, 20, due Feb. 24

Let M_α be a directed family of A-modules with

$\phi_{\beta\alpha}: M_\alpha \longrightarrow M_\beta$, $\phi_\alpha : M_\alpha \hookrightarrow \amalg M_\alpha$

$$\lim = \frac{\amalg M_\alpha}{(\phi_\alpha(m_\alpha) - \phi_\beta(\phi_{\beta\alpha}(m_\alpha)))}$$

so that maps agree

Properties: 1) $\forall y \in \varinjlim_\alpha M_\alpha$, $\exists \beta$ and $x_\beta \in M_\beta$ with $y = \phi_\beta(x_\beta)$

Since $y = \sum_{\alpha \in S, \text{finite}} \phi_\alpha(m_\alpha)$, choose $\gamma \geq$ all $\alpha \in S$

$$\phi_\alpha(m_\alpha) = \phi_\gamma(\phi_{\gamma\alpha}(m_\alpha)) \in \varinjlim M_\alpha$$

$$y = \sum \phi_\alpha(m_\alpha) = \sum_\alpha \phi_\gamma(\phi_{\gamma\alpha}(m_\alpha)) = \phi_\gamma\Big(\sum_\alpha \phi_{\gamma\alpha}(m_\alpha)\Big)$$

2) If $[\phi_\alpha(m_\alpha)] = 0$ in $\varinjlim M_\alpha$, then $\exists \gamma : \phi_{\gamma\alpha}(m_\alpha) = 0 \in M_\gamma$

(If m_α goes to 0 in limit, it goes to 0 in one of the M's)

Since $\phi_\alpha(m_\alpha) = \sum_\delta \phi_\delta(m_\delta) - \phi_\beta(\phi_{\beta\delta}(m_\delta)) = 0$, and choose $\gamma \geq$ all δ, β

$$\phi_{\gamma\alpha}(m_\alpha) = \sum_\delta \phi_{\gamma\delta}(m_\delta) - \underbrace{\phi_{\gamma\beta}(\phi_{\beta\delta}}_{\text{equ} \to \phi_{\gamma\delta}}(m_\delta)) = 0$$

so

$$\phi_\gamma(\phi_{\gamma\alpha}(m_\alpha)) = \phi_\gamma\Big(\sum_\delta \phi_{\gamma\delta}(m_\delta) - \phi_{\gamma\beta}(\phi_{\beta\delta}(m_\delta))\Big)$$ in direct sum

Restatement: If $M_\alpha \longrightarrow \varinjlim M_\alpha$ — this map is ϕ_α by def.

$m_\alpha \longmapsto \phi_\alpha(m_\alpha) = 0$, then since $M \longrightarrow \amalg M_\alpha$ is injective, $m_\alpha \in \operatorname{Ker}(\amalg M_\alpha \longrightarrow \varinjlim M_\alpha)$. Since

$$\begin{tikzcd} M \arrow[r, "\phi_\alpha"] \arrow[dr, "\tilde{\phi}_\alpha"'] & \amalg M_\alpha \arrow[d] \\ & \varinjlim M_\alpha \end{tikzcd}$$

Ker is generated by things of form $\phi_\delta(m_\delta) - \phi_\beta(\phi_{\beta\delta}(m_\delta))$,

$$\phi_\alpha(m_\alpha) = \sum_{\substack{\text{some pairs}\\(\delta,\beta)}} (\phi_\delta(m_\delta) - \phi_\beta(\phi_{\beta\delta}(m_\delta))) \in \underset{\text{not lim}}{\underline{\underline{\amalg M_\alpha}}}$$. Choose

$\gamma \geq \delta, \beta, \alpha$. Claim: $\phi_{\gamma\alpha}(m_\alpha) = 0$.

Give up!

$\varinjlim M_\alpha$ with $\phi_\alpha : M_\alpha \to \varinjlim M_\alpha$ satisfies

$$\mathrm{Hom}_A(\varinjlim M_\alpha, N) \simeq \{ f_\alpha \in \mathrm{Hom}(M_\alpha, N) \}$$

$f_\alpha = f \circ \phi_\alpha$

with
$$\begin{tikzcd} M_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "\phi_{\beta\alpha}"'] & N \arrow[d, "f_\beta"] \\ & M_\beta \end{tikzcd}$$

$$\begin{tikzcd} M_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, "\phi_{\beta\alpha}"'] & \varinjlim M_\alpha \arrow[d] \\ M_\beta \arrow[ur, "\phi_\beta"] & N \end{tikzcd}$$

Any $\{f_\alpha\} \in \mathrm{Hom}(M_\alpha, N)$,

get unique $f : \mathrm{Hom}_A(\coprod M_\alpha, N)$

$f(\phi_\alpha(M_\alpha) - \phi_\beta(\phi_{\beta\alpha}(M_\alpha))) = 0$

$= f_\alpha(m_\alpha) - f_\beta(\phi_{\beta\alpha}(m_\alpha))$

So any $\overset{\text{①}}{\varinjlim M_\alpha}$ $\overset{\text{②}}{\varinjlim M_\alpha}$ are canonically isomorphic

$$\begin{tikzcd} \overset{\text{①}}{\lim M_\alpha} \arrow[r, shift left, "T"] & \overset{\text{②}}{\lim M_\alpha} \arrow[l, shift left, "U"] \\ M_\alpha \arrow[u, "\psi_\alpha"] \arrow[ur, "\phi_\alpha"'] & \end{tikzcd}$$

with $\phi_\beta \phi_{\beta\alpha} = \phi_\alpha$

$\psi_\beta \phi_{\beta\alpha} = \psi_\alpha$

Since lim has universal mapping property, there are $T : ① \to ②$, $U : ② \to ①$, such that the diagram commutes. Thus $T \circ \psi_\alpha = \phi_\alpha$, $U \circ \phi_\alpha = \psi_\alpha$, so $TU = id$, so by symmetry $TU = UT = id$

Localization

Easy generalization of $\mathbb{Z} \to \mathbb{Q}$: A int domain, S mult closed subset with $1 \in S$, $0 \notin S$, take same construction with $\langle a, s \rangle$, get subring of quotient field A_S.

Now let A ring, S mult. closed. A_S = set of equivalence classes $\langle a, s \rangle$ with $\langle a, s \rangle \sim \langle b, t \rangle \iff \exists\, u \in S$ with $u(at - bs) = 0$.

Transitivity: $\langle a, s \rangle \sim \langle b, t \rangle$, $\langle b, t \rangle \sim \langle c, u \rangle \Rightarrow \langle a, s \rangle \sim \langle c, u \rangle$

$v(at - bs) = 0 \quad w(ct - bu) = 0$

$wuv(at - bs) = 0 \quad vws(ct - bu) = 0$

$wuvat = wuvsb = vwsbu = vwsct$

$\underset{\in S}{vwt}(au - cs) = 0$

2/12/09

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \\
A_S \arrow[ur, "f_S"'] &
\end{tikzcd}$$

If $f(S) \subseteq B^{\times}$, f factors uniquely through A_S

$f: A \to B$, $f(s)$ unit, then $f(a/s) = f(a)f(s)^{-1}$

If $a/s = b/t$, $u(ta-bs)=0$, so $f(a)f(s)^{-1} = f(b)f(t)^{-1}$

$f(u(ta-bs))=0$

$f(uta)=f(ubs)$

$f(u)f(t)f(a) = f(u)f(s)f(b)$

$f(t)f(a) = f(s)f(b)$

$f(s)^{-1}f(a) = f(t)^{-1}f(b)$

Similarly for modules

M_S is an A_S-mod

$M_S \overset{\text{canonically}}{\cong} A_S \otimes_A M$

$am/s \longmapsfrom a/s \otimes m$

$m/s \longmapsto 1/s \otimes m$

If $m/s = n/t$, does $\frac{1}{s} \otimes m = \frac{1}{t} \otimes n$

$u(tm-sn)=0$

$1 \otimes utm = 1 \otimes usn$

$ut \otimes m = us \otimes n$

$\frac{1}{uts}(ut \otimes m) = \frac{1}{uts}(us \otimes n)$

$\frac{1}{s} \otimes m = \frac{1}{t} \otimes m$

$B \otimes_A M$ is a B-module by

$b(b' \otimes m) = bb' \otimes m$

Localization is exact

$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ exact $\Rightarrow$

$(M_1)_S \xrightarrow{f_S} (M_2)_S \xrightarrow{g_S} (M_3)_S$ exact

$x/s \longmapsto 0 \in (M_3)_S$ $\quad x \in M_2$

$g_S(x/s) = g(x)/s$

$g(x)/s = 0/1$ in $(M_3)_S \Rightarrow t(1 \cdot g(x) - 0 \cdot s) = 0$

$tg(x) = 0$

$g(tx) = 0 \quad tx = f(y)$

$x/s = \frac{1}{st} \cdot tx$

$f_S(y/st) = f(y)/st = \frac{tx}{st} = x/s$

Conclusion: A_S is a flat A-module

Lecture 7 (2009-02-19)

2/19/09

A ring, I directed set,
M_i, $i \in I$ a directed system of A-modules

$M = \varinjlim_{i\in I} M_i \quad \phi_{ji} : M_i \to M_j \,, \quad \phi_i : M_i \to M$

Prop. $x \in M_i$, $\phi_i(x) = 0 \Rightarrow \exists j \geq i : \phi_{ji}(x) = 0$

Let S be directed set, $k \in S$, $\exists k \geq i \; \forall i \in S$, $\underset{i\in S}{M_i}$ directed system, then
$M_k \xrightarrow{\phi_k} M = \varinjlim_{i \in S} M_i$, ϕ_k is an isomorphism
This is because M_k satisfies the universal mapping property

Let T be a finite subset of I with a maximum element in T.

$M = \bigoplus_{i\in I} M_i \big/ (\text{relations}) \leftarrow$ of the form $\langle \phi_{ji}(x_i)\rangle - \langle x_i \rangle \quad = \bigoplus_{i\in I} M_i / R$

Let $M_T = \dfrac{\bigoplus_{i\in T} M_i}{\underset{i,j\in T}{\langle \phi_{ji}x_i\rangle - \langle x_i\rangle}} \xrightarrow{\psi_T} M$, i.e. $\bigoplus_{i\in T} M_i / R_T \longrightarrow \bigoplus_{i\in I} M_i / R$

Fact: $R = \bigcup_T R_T$.

If $x \in M_i$ $\phi_i(x) = 0$ in M, then $\langle x \rangle = \overset{\text{a relation}}{r} \in R$. So $\exists T$, with $i \in T$, so that $r \in R_T$. Thus $M_K \simeq M_T = \bigoplus_{i\in T} M_i / R_T$
$x \longmapsto 0 \in M_T \simeq M_K$
The maximum element of T is the j asked for in theorem.

Localization: ① A ring, P prime ideal, $S = A - P$. S has 1, is multi. closed since $s_1 \notin P$, $s_2 \notin P \Rightarrow s_1 s_2 \notin P$ (contrapositive of prime ideal) $A_P = A_S$

② $f \in A$, let $S = \{f^n, n \geq 0\}$ we also write $A_f = A_S$

For an alg. variety over k,
$A = k[t_1, \dots, t_r] = \dfrac{k[T_1, \dots, T_r]}{(f_1, \dots f_s)}$, m a maximal ideal,
$\exists \phi : A \longrightarrow k$ with $\ker(\phi) = m$, $\phi(T_i) = t_i$, $f_j(t_1, \dots, t_r) = 0$
A_P is a local ring with max ideal PA_P

M an A-module,

TFAE: ① $M=0$

② $M_p = 0$ $\forall$ prime ideals p

③ $M_m = 0$ $\forall$ max ideals m

Pf. ① $\Rightarrow$ ② $\Rightarrow$ ③. Suppose ③ and $M \neq 0$. Let $x \neq 0 \in M$.

Let $Ann(x) = \{a \in A : ax = 0\}$, $Ann(x)$ is an ideal in A, $\neq A$; $\exists$ max ideal m : $Ann(x) \subseteq m$, $x \neq 0 \in M_m$ because $m/s = 0 \Leftrightarrow \exists t \in S$: $tm = 0$, so $M_m \neq 0$

Cor. M, N two A-modules, $f: M \to N$ an A-hom

TFAE ① f is injective (sur) (bij)

② $f_p : M_p \to N_p$ is injective $\forall$ prime p

③ $f_m : M_m \to N_m$ is " $\forall$ max m

M_S is a functor, $f: M \to N$ becomes

$f(m/s) = f(m)/s$ $M_S \to N_S$

Look at
$$0 \to Ker(f) \to M \xrightarrow{f} N \to Coker(f) \to 0$$

$$0 \to (Ker(f))_p \to M_p \xrightarrow{f_p} N_p \to (Coker(f))_p \to 0$$

$$0 \to (Ker(f))_m \to M_m \xrightarrow{f_m} N_m \to (Coker(f))_m \to 0$$

If f_p inj. for all p, then $(Ker(f))_p = 0$ $\forall$ prime p, so $Ker(f) = 0$, so f inj. — rest are similar.

TFAE ① M flat A-mod

② M_p flat A_p-mod $\forall p$

③ M_m flat A_m-mod $\forall m$

Fact: For M an A-mod, B an A-alg (so $f: A \to B$)

$N = B \otimes_A M$ is a flat B-module

Pf. Consider $0 \to R_1 \to R_2 \to R_3 \to 0$

We can write N also as $M \otimes_A B$

So is $0 \to (M \otimes_A B) \otimes_B R_1 \to (M \otimes_A B) \otimes_B R_2 \to (M \otimes_A B) \otimes_B R_3 \to 0$ exact?

$\wr\|$

$0 \to M \otimes_A R_1 \to M \otimes_A R_2 \to M \otimes_A R_3 \to 0$

with viewing R_i as A-mods via $f: A \to B$

2/19/09

$(M \otimes_A B) \otimes_B R \simeq M \otimes_A R$

$m \otimes b \otimes r \longmapsto m \otimes br$

$(m \otimes 1) \otimes r \longleftarrow\!\shortmid\; m \otimes r$

we should really not define maps on tensors; we should define multilinear map, show well-def.

$(M \times R) \longrightarrow (M \otimes_A B) \otimes_B R$

$(m, r) \longmapsto (m \otimes 1) \otimes r$

$(m_1 + m_2, r) \longmapsto [(m_1 + m_2) \otimes 1) \otimes r$

$\shortparallel$

$(m_1 \otimes 1) \otimes r + (m_2 \otimes 1) \otimes r$

can't define maps out on tensors, but as outputs they're okay

$(am, r) \longmapsto (am \otimes 1) \otimes r = a(m \otimes 1) \otimes r = a(m \otimes 1 \otimes r)$

The trilinear map in other direction is tougher...

Prop. Prime ideals of A_S are in bijection with prime ideals P of A with $P \cap S = \emptyset$.

$P \subseteq A \rightsquigarrow PA_S$

reverse: $Q \subseteq A_S \rightsquigarrow \varphi^{-1}(Q)$ where $\varphi : A \to A_S$

because if x/s, for $x \in P$, $s \in S$,

$x/s \; y/t = z/u \in PA_S$

$x, y \in A$, $s, t, u \in S$, $z \in P$

so $uxy = stz \in P$, $u \notin P$ so $x \in P$ or $y \in P$, so x/s or $y/t \in PA_S$

Cor. Prime ideals of A_P are in 1-1 correspondence with prime ideals of $A \subseteq P$

Def. A ring A is Noetherian if every ascending chain of ideals stops.

$I_1 \subseteq I_2 \subseteq \ldots \Rightarrow \exists n : I_n = I_{n+1} = \ldots$

Prop. ① Any collection of ideals has a maximum element

② Every ideal is finitely generated

① Easy; ② If I ideal of A, $I = 0$ ✓

if not $x_1 \in I \neq 0$, $I = (x_1)$ ✓

if not $x_2 \in I - (x_1)$

$\langle x_1 \rangle, \langle x_1, x_2 \rangle, \langle x_1, x_2, x_3 \rangle, \ldots$

$I = \bigcup_{n=1}^{\infty} I_n \quad I = (f_1 \ldots f_r)$

so $\exists n : f_1 \ldots f_r \in I_n$, so $I_n = I$

Examples

fields, PIDs,

$\mathbb{Z}$, $k[T]$, $\mathbb{Z}[i]$

Not Noetherian:

$k[T_1, \ldots \longrightarrow]$

algebraic integers

Thm: ① A noeth $\Rightarrow A[T]$ noeth (Hilbert Basis Thm)

② A noeth $\Rightarrow A/I$ noeth

③ ring of integers in a number field is noetherian

④ A noeth $\Rightarrow A_S$ noeth

Lecture 8 (2009-02-24)

2/24/09

A Noetherian ring has

1) Every ascending chain of ideals stops
2) Every nonempty set of ideals has a maximal element. } equiv just as statement of ordered sets
3) Every ideal is f.g.

A Noetherian module has

1)
2) submodule replaces ideal
3)

Prop. If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact sequence of A-modules,

M noetherian $\Leftrightarrow$ M', M'' noetherian

($\Rightarrow$)

Pf. $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n$ asc. chain in M'' so

$\{g^{-1}(M_i)\}$ asc. chain in M, so stops, so $M_i = g(g^{-1}(M_i))$

We're acting like f is an inclusion

An asc. chain in M' is a chain in M (under f) and so stops

($\Leftarrow$) M', M'' noeth.

$\{M_n\}$ asc. chain in M, then $g(M_n)$ asc chain $\subseteq M''$, so stops

$M_n \cap M'$ asc chain in M', so stops.

To prove: $R \subseteq S$ submodules of M,

$g(R) = g(S)$ and $R \cap M' = S \cap M' \Rightarrow R = S$

$x \in S$; $g(x) = g(y)$, for $y \in R$, so $g(x-y) = 0$, so $x - y \in M'$,

so $x - y \in S \cap M'$, so $x - y \in R \cap M' \subset R$, so $(x-y) + y \in R$

Cos. M, N noeth $\Rightarrow$ $M \oplus N$ noeth

M_i noeth $\Rightarrow$ $\bigoplus_{i=1}^{n} M_i$ noeth

A^n is noeth if A is noeth

Thm. A noeth, M an A module, then M is noeth $\Leftrightarrow$ M is f.g.

Pf. $\Leftarrow$ $\exists$ surjection $A^n \to M \to 0$

Cor. A noeth $\Rightarrow$ A/I noeth

Cor. A noeth, B f.g. A-module $\Rightarrow$ B noeth (just special case)
an A algebra which is also

Note: A noeth $\not\Rightarrow$ subring of A noeth.

Ex. $A = K(x_1, \ldots)$, subring $= K[x_1, \ldots]$

Prop. A noeth $\Rightarrow$ A_S noeth

Check $I \subseteq A_S$ $f: A \to A_S$

Prove $f(f^{-1}(I)) = I$

Hilbert Basis Thm. A noeth $\Longrightarrow$ $A[x]$ noeth

Pf. Let I be ideal of $A[x]$. for any $f \in I$, let $a(f)$ be the leading coefficient of f (Define $a(0) = 0$). Claim: $\{a(f)\}$ form ideal in A, we have $a(cf) = c\,a(f)$, If $\underset{J}{\shortparallel}$

$$f = a_n x^n + \cdots a_0$$

$$g = b_m x^m \cdots \quad , m \leq n \text{ wlog}$$

$a(x^{n-m} g) = b_n$ so $a(f + x^{n-m} g) = a_n + b_m$ or $a_n = -b_m$.
So $\exists f_1, \ldots f_n \in A[x]$: $J = (a(f_1), \ldots a(f_n))$. Let f_i have degree r_i, Let $r = \max\{r_i\}$, Let $h \in I$ have $\deg(h) \geq r$, so $h = b_s x^s + \ldots$ $b_s \in J$, so $b_s = \sum c_i a(f_i)$ so $\deg(h - \sum c_i x^{s-r_i} f_i) < \deg(h)$. Thus $h = \underbrace{\sum_{i=1}^{n} h_i f_i}_{\in I} + \underset{\in I}{p}$, $\deg(p) < r$, so $p \in I \cap \{\text{poly of deg} < r\}$ $\leftarrow$ which is f.g. A-module, call it M

generated by $1, x, \ldots, x^{r-1}$. $I \cap M$ is f.g. A mod, generated by $p_1, \ldots, p_k$ so $f_1, \ldots f_n, p_1, \ldots p_k$ generate I.

Cor. A noeth $\Rightarrow$ $A[X_1, \ldots X_n]$ is noeth,

Cor. A noeth $\Rightarrow$ $A[x_1, \ldots, x_n]$ is noeth

X denotes variables

x denotes objects, could be numbers

In particular interested in $A = k$, $A = \mathbb{Z}$.

Prop. $A \subseteq B \subseteq C$, A noeth, C is f.g. A-alg. and C is either an f.g. B-module or integral over B. Then B is an f.g A-algebra.

(this helps in Hilbert Nullstellensatz)

Def. $A \subseteq B$, $x \in B$ is integral over A if $\exists$ elements $a_{n-1}, \ldots a_0 \in A$

$$x^n + a_{n-1} x^{n-1} + \cdots a_0 = 0$$

2/24/09

integral over ring is like algebraic over field

Prop. $x \in \mathbb{Q}$, x integral/$\mathbb{Z}$ $\Rightarrow$ $x \in \mathbb{Z}$

Prop. $A \subseteq B$, $x \in B$, TFAE:

1) $x \in B$, integral/A
2) $A[x]$ is an f.g. A-module
3) $A[x] \subset C \subset B$, C is an f.g A-module
4) $\exists$ faithful $A[x]$-module M, which is f.g. as an A-module.

Def. R ring, M R-module, M faithful if $xM = 0 \Rightarrow x = 0$, or equivalently $R \to \mathrm{End}(M)$ is injective

$2 \Rightarrow 3$ (take $C = A[x]$), and $3 \Rightarrow 4$ (take $M = C$); $1 \Rightarrow 2$ because induction, $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, claim: $A[x]$ generated by $1, x, \ldots x^{n-1}$. $x^n = -a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$

$$x^{n+1} = -a_{n-1}x^n - \cdots - a_0 x$$

Now we need $4 \Rightarrow 1$. This is from Nakayama's lemma on $\phi: M \to M$ being multiplication by x. M is f.g. A-module, I ideal of A, $\phi: M \to M$ an A-mod hom, $\phi(M) \subseteq IM$, then ϕ satisfies an equation (in $\mathrm{End}(M)$) of the form

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots a_1\phi + a_0 = 0, \quad a_i \in A.$$

(think should be I)

Take $I = A$, $M = M$, ϕ = multiplication by x

Corollary. $x_i \in B$ integral/A $\Rightarrow$ $A[x_1, \ldots x_n]$ is an f.g.-A-module

Corollary. x, y integral, so $A[x, y]$ f.g. A-module $\Rightarrow$ $x \pm y$, xy integral/A.

Let $\bar{A} = \{x \in B : x \text{ integral}/A\}$. The $\bar{A}$ is an A-algebra, $A \subseteq \bar{A} \subseteq B$, $\bar{A}$ = integral closure of A in B. If $\bar{A} = A$, A is said to be integrally closed in B. If A is integral domain with fraction field K, A is normal (integrally closed) if $\bar{A}$ in K is A.

$[F:\mathbb{Q}] < \infty$, $F = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$

Let $\mathcal{O}_F$ = ring of integers of F = integral closure of $\mathbb{Z}$ in F

$F = \mathbb{Q}(i)$, $\mathcal{O}_F = \mathbb{Z}[i]$

$F = \mathbb{Q}(\sqrt{-3})$, $\mathcal{O}_F = \mathbb{Z}[\frac{-1+\sqrt{3}}{2}]$

Lecture 9 (2009-02-26)

2/26/09

$A^M \longrightarrow B^N$

$(M \otimes_A N) \simeq (M \otimes_A B) \otimes_B N$

$(m \otimes n) \mapsto (m \otimes 1) \otimes n$

"Five Lemma" (easy form)

$$\begin{tikzcd}
 & \bar{u} & & & & \\
 & u \arrow[r, mapsto] \arrow[u, bend left] & y \arrow[r, mapsto] & x \arrow[r, mapsto] & 0 & \\
\text{If} & M_1 \arrow[r] \arrow[d, "f_1"'] & M_2 \arrow[r] \arrow[d, "f_2"'] & M_3 \arrow[r] \arrow[d, "f_3"'] & M_4 \arrow[r] \arrow[d, "f_4"'] & M_5 \arrow[d, "f_5"'] & \text{exact} \\
 & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5 & \text{exact} \\
 & w \arrow[r, mapsto] & z \arrow[r, mapsto] & 0 & 0 &
\end{tikzcd}$$

$\Big\}$ A-modules

f_1, f_2, f_4, f_5 isomorphisms $\Rightarrow$ f_3 isomorphism

Pf. f_3 injective: use f_1 surjective, f_4 injective, f_2 injective

let $y' = y - \bar{u}$; $y' \mapsto x$ $f_2(y') = f_2(y) - f_2(\bar{u}) = z - z = 0$

f_3 surjective: use f_2, f_4 surjective, f_5 injective

Pf. Thm. true for $M = A$

$$\begin{matrix} A \otimes_A N & \simeq & (A \otimes_A B) \otimes_B N \\ \wr| & & \wr| \\ N & & B \otimes_B N \simeq N \end{matrix}$$

Thm. true for $M = A^I$ (finite or infinite direct sum of A's)

True for any A-module; since we can write for any M, $A^I \to M \to 0$. Then write $0 \to P \to A^I \to M \to 0$ for some P; we can similarly do $A^J \to P \to 0$. Thus we get an exact sequence $A^J \to A^I \to M \to 0$. write

$$\begin{tikzcd}
A^J \otimes N \arrow[r] \arrow[d, no head] & A^I \otimes N \arrow[r] \arrow[d, no head] & M \otimes N \arrow[r] \arrow[d, Rightarrow] & 0 \arrow[r] & 0 \arrow[d, no head] \\
(A^J \otimes B) \otimes N \arrow[r] & (A^I \otimes B) \otimes N \arrow[r] & (M \otimes B) \otimes N \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

$\Big\}$ exact because tensor product always exact; to write initial 0's we'd need N to be flat

Cor. $A \subseteq B \subseteq C$, B integral/A, C integral/B

Pf. $x \in C$: $x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 = 0$

let $B' = A[b_0, \ldots, b_{n-1}]$; $B'[x]$ is a f.g. A-module. Thus x integral/A.

$\Rightarrow B'[x]$ f.g. B-module. Thus x integral/B'.

Thm. If $A \subseteq B \subseteq C$, A noeth, C is an f.g. A-algebra;
and C is (f.g. B-module) or (integral over B) (these are the same)
then B is an f.g. A-algebra. (we'll use f.g. B-module)

Pf. $x_1, \ldots x_m$ generate C as A-algebra.
$y_1, \ldots y_n$ generate C as B-module
$\Rightarrow$ a) $x_i = \sum b_{ij} y_j$
b) $y_i y_j = \sum b_{ijk} y_k$
let $B_0 = A[b_{ij}, b_{ijk}]$, so B_0 is noeth;

Any element in C is a polynomial in the x_i's with coeff's in A. Using a) + b), $x \in C$ is a finite comb. of y_i with coefficients in B_0 (by reducing each $y_i y_j$ and $y_i y_l y_m$ to linear comb of y's) Thus C is an f.g. B_0-module. $B \subseteq C \Rightarrow B$ is a f.g. B_0-module since B_0 noeth (?). $\Rightarrow B$ is f.g. A-algebra.

(Also, $A \subseteq B \subseteq C$, B fg A-alg, C fg B-alg, then C fg A-alg)

Prop. k field, E f.g. k-alg, then E field $\Rightarrow E$ is a finite extension of k.

Pf. Let $E = k[x_1, \ldots, x_n]$ for objects x_i. By renumbering, assume $x_1, \ldots, x_r$ alg independent, and $x_{r+1}, \ldots, x_n$ alg. dependent on $x_1, \ldots x_r$. Let $F = k(x_1, \ldots, x_r)$. E is a finite extension of F, so f.g. as F-module. $k \subseteq F \subseteq E$; thm $\Rightarrow$ F f.g. k-algebra. Since $r \geq 1$, Suppose $k(x_1, \ldots x_r) = k[f_i/g_i]$ for $i = 1, \ldots, m$. Take h irred., h not dividing any g_i (exists by looking at $g_1 \cdots g_m + 1$); then $\frac{1}{h} \notin k[f_i/g_i]$. But this can't happen in a field.

($\swarrow$ If $k = \overline{k}$, this forces $A/m \simeq k$) 2/26/09

Weak Nullstellensatz. Let K be a field, A an f.g. K-alg, M a maximal ideal of A. Then A/m is a finite extension of k. (We can conclude this from what we've done)

Strong Nullstellensatz. k field, $A = k[x_1, \dots, x_n]$. k alg closed, I ideal of A, and define $V(I) = \{(\alpha_1, \dots, \alpha_n) \in k^n : f(\alpha_1, \dots, \alpha_n) = 0 \ \forall f \in I\}$

If $S \subseteq k^n$ any subset, with elements $\overline{s} = (s_1, \dots, s_n)$, and define $\mathscr{I}(S) = \{f \in A : f(\overline{s}) = 0 \ \forall \overline{s} \in S\}$.

THEN. $\mathscr{I}(V(I)) = r(I) = \{f \in A : \exists n : f^n \in I\}$.

$\mathscr{I}(V(I)) \supseteq r(I)$ trivial.

Take $f \notin r(I)$. Since $r(I) = \bigcap_{\text{prime } P,\ P \supseteq I} P$, $\exists P$ prime, $P \supseteq I$, with $f \notin P$. Let $B = A/P$, an integral domain. Let $\overline{f}$ = image of f in B. $\overline{f} \neq 0$ since $f \notin P$, so we can take $C = B[1/\overline{f}]$. C is still an f.g. k-alg; let m be maximal ideal of C. By weak Nullstellensatz, $C/m \simeq k$. We have:

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[ddr, bend left=60, "x_i \mapsto \alpha_i"] & B \arrow[d, hook] \\ & C \arrow[d] \\ k \arrow[uu] \arrow[r, "\sim"'] & k \end{tikzcd}$$

$x_i \mapsto \alpha_i \in C/m = k$

Claim: If $(\alpha_1, \dots, \alpha_n) = \overline{\alpha}$ is a point in k^n, need $\overline{x} \in V(I)$, $f(x) \neq 0$.

$g \in I \Rightarrow g(\alpha_1, \dots, \alpha_n) = 0$. $g \in I \Rightarrow g \in P$, so goes to 0 in B.

Def. An algebraic set in k^n is $V(I)$ for some I (when $k = \overline{k}$)

1-1 correspondence $I \rightsquigarrow V(I)$, $S \rightsquigarrow \mathscr{I}(S)$ between radical ideals and algebraic sets and prime ideals and algebraic varieties

Lecture 10 (2009-03-03)

3/3/09

Snake lemma

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0 \end{tikzcd}$$

M's, N's all modules over A

A could be non-commutative, so let M, N all be <u>Left</u> A-Modules.

$\exists$ natural exact sequence $0 \to \mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \to \mathrm{Coker}(f_1) \to$
$\mathrm{Coker}(f_2) \to \mathrm{Coker}(f_3) \to 0$

$$\begin{tikzcd} & \mathrm{Ker}(f_1) \arrow[r] \arrow[d, hook] & \mathrm{Ker}(f_2) \arrow[r] \arrow[d, hook] & \mathrm{Ker}(f_3) \arrow[d, hook] & & \\ 0 \arrow[r] & M_1 \arrow[r, "i"] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 & \text{exact} \\ 0 \arrow[r] & N_1 \arrow[r, "j"] & N_2 \arrow[r] & N_3 \arrow[r] & 0 & \text{exact} \end{tikzcd}$$

If $f_3(x) = 0$, $y \mapsto x$, $f_2(y) = j(z)$

$y' = y + i(w)$ $w \in M_1$

$$\begin{array}{ccc} f_2(y') = & f_2(y) + & f_2(i(w)) \\ \parallel & & \parallel \\ j(z') & & j(f_1(w)) \end{array}$$

$j(z') = j(z) + j(f_1(w))$

$z' = z + f_1(w)$ so $[z'] = [z]$ in $N_1/f_1(M_1)$

$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots M_1 \to M_0$

$\cdots \to N_n \xrightarrow{e_n} N_{n-1} \xrightarrow{e_{n-1}} \cdots N_1 \to N_0$

are complexes if "$d^2 = 0$" i.e. $d_{n-1} \circ d_n = 0$
"$e^2 = 0$" i.e. $e_{n-1} \circ e_n = 0$

A map of complexes is $\{f_n : M_n \to N_n : e_n f_n = f_{n-1} d_n\}$

Def. f_n is homotopic to g_n if $\exists$ maps $\lambda_n : M_n \to N_{n+1}$: $f_n - g_n = e_{n+1}\lambda_n + \lambda_{n-1} d_n$
we say $f \sim g$.

Prop. $\sim$ is equivalence relation.

Prop. Homotopic maps induce the same map on homology.
$h_n(M_\times) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$ $h_n(N_\times) = \mathrm{Ker}(e_n)/\mathrm{Im}(e_{n+1})$

P a projective A-module if $M_1 \twoheadrightarrow M_2 \Rightarrow \mathrm{Hom}(P, M_1) \twoheadrightarrow \mathrm{Hom}(P, M_2)$

Also, a free A-module is projective

Equivalently, If
$$\begin{tikzcd} M_1 \arrow[d] \\ M_2 \arrow[d] \\ 0 \end{tikzcd}$$
, and $P \to 0$, we can lift
$$\begin{tikzcd} P \arrow[r, dashed] \arrow[rdd] & M_1 \arrow[d] \\ & M_2 \arrow[d] \\ & 0 \end{tikzcd}$$

(~~*~~) If an infinite complex $\to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ has

all P_i projective

*) is exact then we say it is a projective resolution of M.

<u>Prop</u>. Projective resolutions always exist
(just take free module on elements)

<u>Thm</u>. Projective resolutions are unique up to homotopy, i.e.

if $\dot{P} \to M$, $\dot{Q} \to M$, $\exists$ maps $f_\cdot : P_\cdot \to Q_\cdot$, $g_\cdot : Q_\cdot \to P_\cdot$

such that $g_\cdot \circ f_\cdot \sim id$, $f_\cdot \circ g_\cdot \sim id$.

Lemma.
$$\begin{tikzcd}
\arrow[r] & P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d] & 0 \\
\arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

$\alpha \in Hom(M, N)$. Suppose $P_\cdot$ is projective complex, and $Q_\cdot$ is just exact. a) Can extend α to a map $f_\cdot$ of complexes and b) $f_\cdot$ unique up to homotopy

$$\begin{tikzcd}
P_1 \arrow[r, "d_1"] \arrow[d, dashed] & P_0 \arrow[r, "d_0"] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
Q_1 \arrow[r, "e_1"] & Q_0 \arrow[r, "e_0"] & N \arrow[r] & 0
\end{tikzcd}$$

we have maps
$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d, dashed] & Ker(d_0) \arrow[d] \\
Q_1 \arrow[r, two heads] & Ker(e_0)
\end{tikzcd}$$
$(Im(e_1) = Ker(e_0))$

$$\begin{tikzcd}
P_2 \arrow[r] & Ker(d_1) \arrow[d] \\
Q_2 \arrow[r, two heads] & Ker(e_1)
\end{tikzcd}$$

$$\begin{tikzcd}
P_2 \arrow[r, "d_2"] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] & P_1 \arrow[r, "d_1"] \arrow[dl, "\lambda_1"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & P_0 \arrow[r, "d_0"] \arrow[dl, "\lambda_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & M \arrow[d, "\alpha"] \\
Q_2 \arrow[r, "e_2"'] & Q_1 \arrow[r, "e_1"'] & Q_0 \arrow[r, "e_0"'] & N
\end{tikzcd}$$

want $f_0 - g_0 = e_1 \lambda_0$. $f_0 - g_0 : P_0 \to Ker(e_0)$

$Q \twoheadrightarrow Ker(e_0)$, lift to λ_0

want $f_1 - g_1 = e_2 \lambda_1 + \lambda_0 d_1$, want λ_1

$e_1(f_1 - g_1 - \lambda_0 d_1) = 0$ $\quad e_1 \lambda_0 d_1 = (f_0 - g_0) d_1$

$f_0 d_1 - g_0 d_1 - (f_0 - g_0) d_1 = 0$

$$\begin{tikzcd}
f_1 - g_1 - \lambda_0 d_1 : P_1 \arrow[r] \arrow[dr, bend right, "\text{lift to } \lambda_1"'] & Ker(e_1) \\
 & Q_2 \arrow[u]
\end{tikzcd}$$

3/3/09

Now suppose

$$\begin{tikzcd} P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] \arrow[d, "id"] & 0 \\ Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

each proj. resolution.

$\exists\, f_\bullet : P_\bullet \to Q_\bullet$ map of complexes extending id unique up to homotopy

$g_\bullet : Q_\bullet \to P_\bullet$

$g_\bullet \circ f_\bullet : P_\bullet \to P_\bullet$ extends id, so $g_\bullet \circ f_\bullet$ homotopic to id_P

Let T be a covariant right-exact additive functor from A-mod to A-mod

$M' \to M \to M'' \to 0$ exact

$\Rightarrow T(M') \to T(M) \to T(M'') \to 0$ exact

Left derived functors $L^p T(M)$

proj resolution: $P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M_0 \to 0$ exact.

$$\begin{tikzcd} T(P_n) \arrow[r, "T(d_{n+1})"'] & T(P_{n-1}) \arrow[r, "T(d_n)"'] & T(P_1) \arrow[r] & T(P_0) \end{tikzcd}$$

is just a complex, not necessarily exact

$$L^p T(M) = Ker(T(d_p)) / Im(T(d_{p+1}))$$

Left derived functors of T. we want to show this is independent of P.

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d, shift right, "f_\bullet"'] & M \\ Q_\bullet \arrow[r] \arrow[u, shift right, "g_\bullet"'] & M \end{tikzcd}$$

$g_\bullet \circ f_\bullet \sim id_P$

$f_\bullet \circ g_\bullet \sim id_Q$

$$\begin{tikzcd} T(P_\bullet) \arrow[d, shift right, "T(f_\bullet)"'] \\ T(Q_\bullet) \arrow[u, shift right, "T(g_\bullet)"'] \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r] \arrow[d, shift right, "g\circ f"'] \arrow[d, shift left, "id"] & P_1 \arrow[r] \arrow[d, shift right, "g\circ f"'] \arrow[d, shift left, "id"] & P_0 \arrow[r] \arrow[d, shift right, "g\circ f"'] \arrow[d, shift left, "id"] & 0 \\ P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0 \end{tikzcd} \rightsquigarrow \begin{tikzcd} T(P_2) \arrow[r, "T(d)"] \arrow[d, shift right, "T(g\circ f)"'] \arrow[d, shift left, "id"] & T(P_1) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & T(P_0) \arrow[d, shift right] \arrow[d, shift left] \\ T(P_2) \arrow[r, "T(e)"'] & T(P_1) \arrow[r] & T(P_0) \end{tikzcd}$$

$g\circ f \sim id$

$g\circ f - id = \lambda d + e\lambda$

because T additive functor $\to$ respects composition $\to$ $T(g\circ f) - T(id) = T(\lambda)T(d) + T(e)T(\lambda)$

Thus $T(g) \circ T(f) \sim id$

Same argument $\to T(f) \circ T(g) \sim id$

Example. N a fixed A mod

$T(M) = M \otimes_A N$

If T is exact, $P_{n+1} \to P_n \to P_{n-1} \to \cdots P_1 \to P_0$

$T(P_{n+1}) \to T(P_n) \to T(P_{n-1}) \to \cdots T(P_1) \to T(P_0)$

$L^0 T(N) = T(M)$

$L^p T(M) = 0$ for $p > 0$ even if T only right exact

because $P_2 \to P_1 \to P_0 \to M \to 0$

$T(P_2) \to T(P_1) \to T(P_0) \to \cancel{T(M)} \to 0$

$\mathrm{Tor}_0^A(M,N) = M \otimes_A N$

Any module is direct limit of its f.g. submodules

So if N flat, then $\mathrm{Tor}_p^A(M,N) = 0$ for $p > 0$, any M

If $A = k$ a field, then all N are flat, so $\mathrm{Tor}_p^k(M,N) = 0$ for $p > 0$ any M, N

First interesting case: $A = \mathbb{Z}$

If M a f.g. $\mathbb{Z}$-mod, $M = \mathbb{Z}^r \oplus$ torsion

M f.g., torsion free $\Rightarrow M = \mathbb{Z}^r$

$M = \varinjlim M_\alpha$ for $\underset{\text{f.g.}}{M_\alpha} \subseteq M$, M_α torsion free, M_α free

$N' \to N \to N''$ exact $\Downarrow$

$N' \otimes M_\alpha \to N \otimes M_\alpha \to N'' \otimes M_\alpha$ exact

Use: $\varinjlim$ of exact sequences is exact

$\varinjlim$ commutes with $\otimes$

$N' \otimes M \to N \otimes M \to N'' \otimes M$

$N' \otimes \varinjlim M_\alpha \to N \otimes \varinjlim M_\alpha \to N'' \otimes \varinjlim M_\alpha$

$\varinjlim (N' \otimes M_\alpha) \to \varinjlim (N \otimes M_\alpha) \to \varinjlim (N' \otimes M_\alpha)$

So this is exact

thus torsion-free $\Rightarrow$ flat, and think about why this implies $\forall \mathbb{Z}$-mod M, N, $\mathrm{Tor}_p^A(M,N) = 0$ for $p > 1$.

Lecture 11 (2009-03-05)

3/5/09

T right exact covariant functor from A-mods to A-mods

$L^pT(M)$ is : take projective resolution of M,

$$\begin{tikzcd} {} \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\ (*) \quad {} \arrow[r] & T(P_{n+1}) \arrow[r, "T(d_{n+1})"] & T(P_n) \arrow[r, "T(d_n)"] & T(P_{n-1}) \arrow[r] & \cdots \arrow[r] & T(P_1) \arrow[r] & T(P_0) \arrow[r] & 0 \end{tikzcd}$$

$$L^pT(M) = h_n(*) = {\ker(T(d_n))}/{\mathrm{Im}(T(d_{n+1}))}$$

proved this was independent of choice of P.

For a exact sequence $0 \to M' \to M \to M'' \to 0$, we construct

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ (1) \ {} \arrow[r] & P'_{n+1} \arrow[r] \arrow[d] & P'_n \arrow[r] \arrow[d] & P'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_1 \arrow[r] \arrow[d] & P'_0 \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\ (2) \ {} \arrow[r] & P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & P'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ (3) \ {} \arrow[r] & P''_{n+1} \arrow[r] \arrow[d] & P''_n \arrow[r] \arrow[d] & P''_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P''_1 \arrow[r] \arrow[d] & P''_0 \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & & 0 & 0 & 0 & \end{tikzcd}$$

proj resolutions

Pick (1) and (3) arbitrarily.

Lemma. $0 \to M \to N \to P \to 0$, P projective $\Rightarrow N \simeq M \oplus P$

because we have

$$\begin{tikzcd} N \arrow[r] & P \arrow[r] & 0 \\ & P \arrow[u, "id"'] \arrow[ul, dashed, "s"] & \end{tikzcd}$$

, $s(P) \simeq P$, and $N \simeq s(P) \oplus M$

So define $P_n = P'_n \oplus P''_n$, and we get all vertical maps

$$\begin{tikzcd} 0 \arrow[d] \\ P'_n \arrow[d] \\ P_n \arrow[d] \\ P''_n \arrow[d] \\ 0 \end{tikzcd}$$

, but not the horizontal map (2). We induct.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ P'_1 \arrow[r] \arrow[d] & P'_0 \arrow[r, "d_0"] \arrow[d] & M' \arrow[r] \arrow[d, "i"] & 0 \\ P'_1 \oplus P''_1 \arrow[r] \arrow[d] & P'_0 \oplus P''_0 \arrow[r, "f_0"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ P''_1 \arrow[r] \arrow[ur, dashed] & P''_0 \arrow[r, "e_0"'] \arrow[d] \arrow[ur, dashed, "\tilde{e}_0"] & M'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd}$$

We want $P'_0 \oplus P''_0 \twoheadrightarrow M$

We define

$$\begin{tikzcd} (x,y) \arrow[r] \arrow[d] & i d_0(x) + \tilde{e}_0(y) \arrow[d] \\ y \arrow[r, mapsto] & e_0(y) \end{tikzcd}$$

Let $X_0 = Ker(d_0)$, $Y_0 = Ker(f_0)$, $Z_0 = Ker(e_0)$;

$$\begin{tikzcd} 0 \arrow[r] & X_0 \arrow[r] & Y_0 \arrow[r] & Z_0 \arrow[r] & 0 \\ & & & P''_1 \arrow[ul] \arrow[u] & \end{tikzcd}$$

by snake lemma.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
(1) & T(P_2') \arrow[r] \arrow[d] & T(P_1') \arrow[r] \arrow[d] & T(P_0') \arrow[r] \arrow[d] & T(M') \arrow[r] \arrow[d] & 0 \\
(2) & T(P_2) \arrow[r] \arrow[d] & T(P_1) \arrow[r] \arrow[d] & T(P_0) \arrow[r] \arrow[d] & T(M) \arrow[r] \arrow[d] & 0 \\
(3) & T(P_2'') \arrow[r] \arrow[d] & T(P_1'') \arrow[r] \arrow[d] & T(P_0'') \arrow[r] \arrow[d] & T(M'') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & \underbrace{0}_{\text{chop off}} &
\end{tikzcd}$$

no 0 here because T only right exact

We get 0's at beginning of other vertical maps because

$0 \to P' \hookrightarrow P \to P'' \to 0$ splits, i.e. $P' \oplus P'' \Rightarrow$

$0 \to T(P') \to T(P) \to T(P'') \to 0$ exact.

Prop. Given exact sequence of complexes $0 \to (1) \to (2) \to (3)$,

we get $h_n(1) \to h_n(2) \to h_n(3) \underset{\text{tough one}}{\to} h_{n-1}(1) \to h_{n-1}(2) \to h_{n-1}(3) \to \cdots$ exact

$L^nT(M') \to L^nT(M) \to L^nT(M'') \to L^{n-1}T(M') \to \cdots$ exact

Everything is the same if T is contravariant left exact

$0 \to M' \to M \to M'' \to 0$

$0 \to T(M'') \to T(M) \to T(M')$

$P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$

$T(M) \to T(P_0) \to T(P_1) \to \cdots T(P_n) \xrightarrow{T(d_n)}$

$R^nT(M) = Ker(T(d_n)) / Im(T(d_{n-1}))$

Fix N

$M \to Hom_A(M,N)$

derived functors

$Ext^n_A(M,N)$

$Ext^0_A(M,N) = Hom_A(M,N)$

$Tor_i^A(M,N)$

$T(M) = M \otimes_A N$ $\quad U(N) = M \otimes_A N$

$P_\bullet \to M$ $\quad Q_\bullet \to N$

$h(T(P_\bullet))$ $\quad h(U(Q_\bullet))$

SAME

Prop. T, $\tilde{T}$ right exact functors; suppose we have functors $L^n(T)$, $\tilde{L}^n(T)$:

a) $L^0(T) = T$, $\tilde{L}^0(T) = T$

b) $0 \to M' \to M \to M'' \to 0$, then

and for $\tilde{L}$

$L^nT(M') \to L^nT(M) \to L^nT(M'') \to L^{n-1}T(M') \to L^{n-1}T(M) \to \cdots \quad L^0T(M'') \to 0$

c) $L^n(P) = 0$, $\tilde{L}^n(P) = 0$ if P projective

induct + Snake lemma,

$0 \to M' \to M \to M'' \to 0$ $\quad Q_\bullet \otimes P$

$0 \to Q_\bullet \otimes M' \to Q_\bullet \otimes M \to Q_\bullet \otimes M'' \to 0$

3/5/09

From last time, $i>0$, then $\mathrm{Tor}_i^{\mathbb{Z}}(M,N)=0$ for any N if M is torsion free.

Cf $0\to K\to F\to M\to 0$, M any $\mathbb{Z}$-module

$$\begin{tikzcd}
\mathrm{Tor}_n^{\mathbb{Z}}(F,N) \arrow[r] \arrow[d, equal] & \mathrm{Tor}_n^{\mathbb{Z}}(M,N) \arrow[r] & \mathrm{Tor}_{n-1}^{\mathbb{Z}}(K,N) \arrow[r] & \mathrm{Tor}_{n-1}^{\mathbb{Z}}(F,N) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

K is torsion-free $n\geq 2$

so $\mathrm{Tor}_{n-1}^{\mathbb{Z}}(K,N)=0$

INJECTIVES

covariant left exact functor : $N \rightsquigarrow \mathrm{Hom}_A(M,N)$

we need injective resolutions

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[dr] & M \arrow[d, two heads] \\
& N
\end{tikzcd}$$

$M\to N\to 0$

$\Rightarrow \mathrm{Hom}(P,M)\to \mathrm{Hom}(P,N)\to 0$

if P projective

so injective is dual:

$$\begin{tikzcd}
N \arrow[r, hook] \arrow[d] & M \arrow[dl, dashed] \\
I &
\end{tikzcd}$$

$0\to N\to M$

$\mathrm{Hom}(M,I)\to \mathrm{Hom}(N,I)\to 0$

injective resolution

$0\to M\to I_0\to I_1 \xrightarrow{d_1} \cdots I_j \to$ exact

T covariant left exact

$R^nT(M) = \mathrm{Ker}(T(d_n))/\mathrm{Im}(T(d_{n-1}))$

independent of choice of injective resolution

cohomological functor

$0\to N'\to N\to N''\to 0$

$N\rightsquigarrow \mathrm{Hom}_A(M,N)$

$0\to \mathrm{Hom}_A(M,N')\to \mathrm{Hom}_A(M,N)\to \mathrm{Hom}_A(M,N'')\to \mathrm{Ext}_A^1(M,N')\to \cdots$

Thm. Given any A, M, exists injective A-module I and an injection $M\to I$,

$0\to M\to I_0\to C_0\to 0$

$0\to C_0\to I_1\to C_1\to 0$

$0\to C_1\to I_2 - C_2\to 0$

Thm. L injective $\iff$ $\forall$ left ideals J of A, and every homomorphism $J\to L$, $\exists x\in L$: $f(\lambda)=\lambda x$ for $\lambda\in J$

$$\begin{tikzcd}
J \arrow[r, hook] \arrow[d, "f"'] & A \arrow[dl, dashed, "g"] \\
L &
\end{tikzcd}$$

Lecture 12 (2009-03-10)

3/10/09

Prop. Let A be local, noeth; M an f.g. A-mod; then TFAE
1) M is free 2) M is flat 3) $\mathrm{Tor}_1^A(M,k)=0$ where $k=$ residue field A/m

1)⇒2), 2)⇒3) ✓ To prove 3)⇒1), note that $M\otimes_A k = M/mM$, pick a basis $\bar{x}_1,\dots,\bar{x}_n$ of $M\otimes_A k$, lift to $x_1,\dots x_n \in M$. Let $A_n \xrightarrow{\phi} M$ with $e_i \mapsto x_i$. We write $0\to \underset{\text{(kernel)}}{K} \to A^n \xrightarrow{\phi} M \to \underset{\text{(cokernel)}}{C} \to 0$

tensor with k; $k^n \to M\otimes_A k \to C\otimes_A k \to 0$, Surjective ⇒ $C\otimes_A k=0$

Nakayama ⇒ $C=0$. So $0\to K\to A^n\to M\to 0$. But then we have

$\mathrm{Tor}_i^A(M,k) \to K\otimes_A k \to k^n \xrightarrow{\sim} M\otimes_A k \to 0$, and $\mathrm{Tor}_1^A(M,k)=0$ by hypothesis,

so $K\otimes_A k=0$ because $k^n \xrightarrow{\sim} N\otimes_A k$, A noeth ⇒ K f.g., Nakayama ⇒ $K=0$,

so $A^n \xrightarrow{\sim} M$.

Corollary. A noeth; M f.g. A-mod; then M flat ⇔ M projective ⇔ M "locally free"

locally free: either (*) or (**)
  (*) M_p free A_p-module for all prime ideals p
  (**) Given a $p \in \mathrm{Spec}(A)$, $\exists f \notin p$: M_f is a free A_f module

Example. If A is Dedekind domain, e.g. $A=O_F$ for number field F, then for I ideal of A, I free ⇔ I principal

Thm. A, not necessarily commutative; M left A-mod ⇒ ∃ injective A-mod I and an inclusion $M \hookrightarrow I$

$$\begin{tikzcd} N \arrow[d, hook, "\beta"'] \arrow[dr] & \\ M \arrow[r, dashed] & I \end{tikzcd}$$

Lemma. I injective ⇔ $\forall J \hookrightarrow A$, J left ideal of A, and
$$\begin{tikzcd} J \arrow[r, hook] \arrow[dr, "f"'] & A \arrow[d, "h"] \\ & I \end{tikzcd}$$,
$\exists x\in I$: $f(\lambda)=\lambda x$. This means we can extend f to $h: A\to I$.
If $h(\lambda)=h(\lambda\cdot 1)=\lambda h(1)=\lambda x$

Pf. Contradiction. Look at all pairs Σ: (N',g), $g: N'\to I$, $N\subseteq N'\subseteq M$, $g|_N=f$
By Zorn's lemma, ∃ maximal pair (N',g)
(define $(N',g')\le(N'',g'')$ if $N'\subseteq N''$ and $g''|_{N'}=g'$)
Suppose $N'\neq M$, and let $x\in M-N'$. Let $J=\{a\in A : ax\in N'\}$, let $f: J\to I$.
$f(\lambda)=g'(\lambda x)$
$\exists v\in I$: $f(\lambda)=\lambda v$, $g'(\lambda x)=\lambda v$. We can now extend
g' to $h: N'+\langle x\rangle \to I$, $\underset{n\in N'}{h(n+ax)}=g'(n)+av$

Suppose $n_1 + a_1 x = n_2 + a_2 x$; need to show $g'(n_1) + a_1(u) = g'(n_2) + a_2 u$

$g'(n_1 - n_2) = (a_2 - a_1) u$

$n_1 - n_2 = (a_2 - a_1) x$

$g'(n_1 - n_2) = g'((a_2 - a_1) x) = (a_2 - a_1) u$

$g'(n_1) - g'(n_2) = a_2 u - a_1 u$

Thus the lemma is proved. We have essentially shown that if I is "Somewhat" injective, i.e. we can extend $J \hookrightarrow A$ to h for when J is an ideal of A, then I is injective.

$$\begin{tikzcd} J \arrow[r, hook] \arrow[dr, "f"'] & A \arrow[d, dashed, "h"] \\ & I \end{tikzcd}$$

Prop. If A is a PID, I injective $\iff$ I is divisible

Def. I is divisible $\iff$ $\forall x \in I$, $\forall a \in A$, $a \neq 0$, $\exists y \in I$: $ay = x$

Example: $A = \mathbb{Z}$, $I = \mathbb{Q}$ or $I = \mathbb{Q}/\mathbb{Z}$

$$\begin{tikzcd} (a) \arrow[r, hook] \arrow[dr, "ba \mapsto bx"'] & A \arrow[d, "1"] \\ & I \end{tikzcd} \qquad y \qquad a \mapsto x = ay$$

If G any cyclic group, $\exists$ non-zero map $G \to \mathbb{Q}/\mathbb{Z} = T$

Let M be left A-module. Let $\hat{M} = \mathrm{Hom}_{\mathbb{Z}}(M, T)$; $\hat{M}$ is a right A-module. If $f \in \hat{M}$, $(fa)(x) = f(ax)$, $(f(ba))(x) = f(bax) = (fb)(ax) = (fa)(b)(x)$

$\hat{\hat{M}} = \mathrm{Hom}_{\mathbb{Z}}(\hat{M}, T)$ is a left A-module. $\begin{matrix} M \to \hat{\hat{M}} \\ x \mapsto g_x \end{matrix}$ is injective ; $x \in M$, $x \neq 0$.

$g_x(f) = f(x)$. Claim: M projective right A-mod $\Rightarrow \hat{M}$ injective left A-mod.

$$\begin{tikzcd} N \arrow[d, hook] \arrow[dr, "f"] & \\ P \arrow[r] & \hat{M} \end{tikzcd} \qquad \hat{f} : \begin{tikzcd} \hat{\hat{M}} \arrow[r] & \hat{\hat{N}} & \\ M \arrow[u] \arrow[ur] \arrow[rr, dashed] & & P \arrow[ul] \end{tikzcd}$$

If Q projective right A-mod $Q \twoheadrightarrow \hat{M}$, then $M \hookrightarrow \hat{\hat{M}} \hookrightarrow \hat{Q}$, so $\hat{Q}$ injective

Corollary

Can define $\mathrm{Ext}^i_A(M, N)$ by taking injective resolution $N \to I_0 \to I_1 \to \cdots I_n \to$,

$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, I_0) \to \cdots$, and taking homology.

G a group, A comm ring, $A[G] = \{ \sum_\sigma a_\sigma \langle \sigma \rangle \}$. If M G-module, M ab group under $+$

G-modules $\to$ ab groups

$$\begin{tikzcd} M \arrow[r, rightsquigarrow] & M^G = \{ m \in M : \sigma m = m \ \forall \sigma \in G \} \arrow[d, phantom, "\parallel"] \\ & \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) \end{tikzcd}$$

Cohomology of gps = derived functors of M^G

$\mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, m) \overset{\mathrm{def}}{=} H^i(G, m)$

Lecture 13 (2009-03-12)

Discrete valuation rings 3/12/09

K field

A subring of K is a valuation ring of K $\Leftrightarrow \forall x \in K$, $x \in A$ or $\frac{1}{x} \in A$.

So $K = ff(A)$

Prop. 1) A local ring

2) $A \subseteq A' \subseteq K$, then A' is a valuation ring

3) A integrally closed in K

Pf. Let m be $= \{x \in A, \frac{1}{x} \notin A\}$

m = set of non units of A

$b \in A$, $x \in m$, $bx \in A$

$\frac{1}{bx} \in A$ $\frac{1}{x} = b \cdot 1/bx \in A$

To show m is ideal,

$x \cdot y \in m \rightarrow x+y \in A$

$(1+xy^{-1})y$

Either xy^{-1} or $x^{-1}y \in A$

Suppoce $xy^{-1} \in A$

$$\underset{\in A}{(1+xy^{-1})}y = y+x$$

m is unique max ideal of A

2 - duh

3 - $x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0$ $a_i \in A$, $x \in K$

$x \notin K \Rightarrow 1/x \in A$

$$x + a_{n-1} + a_{n-2}/x + \dots + a_1/x^{n-2} + a_0/x^{n-1} = 0$$

$$\Rightarrow x = -(a_{n-1} + \dots + a_0/x^{n-1}) \in A$$

Def. Γ ordered ab gp if

a) Γ ab gp

b) Γ totally ordered ($x \le y, y \le z \Rightarrow x \le z$, $x \le y, y \le x \Rightarrow x = y$ $x \le y$ or $y \le x$)

c) $x \ge y \Rightarrow x+a \ge y+a$

Def. let K be field, Γ ordered ab gp.

$v : K \div \{0\} \to \Gamma$ is a valuation if $v(xy) = v(x) + v(y)$

$v(x+y) \geq \min(v(x), v(y))$

A valuation ring of K, let $U = A^*$. Thus $A^* \subseteq K^*$.

Claim: K^*/A^* is an ordered ab gp.

$[x] \geq [y] \iff x = ay \quad a \in A$

$x' = ux \quad x' = uay$

$x' = by \quad x = u^{-1}by$

$x = ay \quad y = bz \Rightarrow x = abz$

$x = ay\,,\ y = bx \Rightarrow x = abx\,,$ so $ab = 1$,

so $a, b \in A^*$

so $[x] = [y]$ in K^*/A^*

$x = ay \Rightarrow$

$bx = aby$

$x/y \in A$ or $y/x \in A$

$x = ay$ or $y = ax$

$v : K^* \to \Gamma = K^*/A^*$

projection

wlog $v(x) \leq v(y)$

$v(x+y) \geq \min(v(x), v(y))$

$y \geq x \quad y = ax$

$x + y = x + ax = (1+a)x$

$x + y \geq x$

If $v : K^* \to \Gamma$ is a valuation

let $A = \{x \in K : v(a) \geq 0\}$

A is a valuation ring

$v(ab) = v(a) + v(b) \geq 0$

$v(a) \geq 0\ ,\ v(b) \geq 0$

$v(a+b) \geq \min(v(a), v(b)) \geq 0$

$A^* = \{a : v(a) = 0\}$

$\mathbb{R}$ is an ordered ab gp under $+$

Any subgroup of $\mathbb{R}$ is too: $\mathbb{Q}$, $\mathbb{Z}$,

"rank one" = subgp of $\mathbb{R}$

$\mathbb{R}^n$ with lexicographic ordering

If $\Gamma = \mathbb{Z}$, and v is onto $K^* \twoheadrightarrow \mathbb{Z}$,

A is a discrete valution ring (DVR)

3/12/09

Exercize: The ideals of A are $\{x : v(x) \geq \alpha\}$ or $\{x : v(x) > \alpha\}$

for $\alpha \in \Gamma$. When is an ideal prime? When is a valution ring noeth?

p prime, $A = \{x = \frac{a}{b} \in \mathbb{Q}\}$ $(a,b) = 1$

such that $p \nmid b$

A is a DVR, $\mathbb{Z}_{(p)}$

$V_p : A \to \mathbb{Z}$ $\quad a = p^k u/b$

$p \nmid u$ unique

$v_p(a) = k$

$x = a/p^k u$

$v(x) = -k$

$v_p : K \to \mathbb{Z}$

$v_p(a/b) = v_p(a) - v_p(b)$

F field, $F(x)$

take q = irred.

$A_q = \{f/g,\ q \nmid g\}$

$f = q^k u$, $v_q(f) = k$.

$F((x))$,

$v(f = a_{-n}x^{-n} + a_{n+1}x^{-n+1} \cdots a_0 + \cdots)$

$a_{-n} \neq 0$, $v(f) = -n$

Dimension theory

A ring, prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ ← this is said to have length n

Def. $\dim(A)$ = Krull dimension = max length of any chain of prime ideals

$\dim(\text{field}) = 0$ $\quad \dim(K[x_1, x_2, \ldots]) = \infty$

$\dim(\text{PID}) = 1$ because no non-maximal prime ideals $\neq (0)$

$0 \subsetneq P_1 \subsetneq P_2 \neq A$ impossible

$(c) \subset (a)$ $\quad$ If $a \in P_1$, $P_1 \neq P_2$ X $\quad c = b_1 a = uac$ $\quad b_1 = ab_2, b_2 \in P_1$

$c = b_1 a$ $\quad b \in P_1$ $\quad 1 = ua$ $\quad (b_1) \subseteq (b_2)$

$b_1 a \in P_1$ $\quad (c) \subsetneq (b_1)$, $b_1 = uc$ $\quad$ a not unit so let $\quad$ we could do this infinitely, so on but PID's are noeth.

Question: Is it true that all maximal chains of prime ideals have same length? No.

A DVR, let $m \subseteq A$ $m = (\pi)$

$A[x]/(\pi x - 1)$

$A[x]$ 2-dim ring

A noeth $\Rightarrow \dim(A[x]) = \dim(A) + 1$

In general, $\dim(A[x]) \geq 1 + \dim(A)$
$\leq 1 + 2\dim(A)$

$0 \subsetneq \underset{\underset{(\pi)}{\|}}{m_A} \subsetneq (m_A, x) = (\pi, x)$

$\underline{\underline{(\pi x - 1)}}$ is a maximal ideal

$A[x]/(\pi x - 1) \to K = ff(A)$

$x \to 1/\pi$ but $K = A[\frac{1}{\pi}]$

this is onto, so $0 \subsetneq (\pi x - 1)$ max chain of length 1

If A is either $F[x_1, \ldots, x_n]$
or $\mathbb{Z}[x_1, \ldots, x_n]$

then all max chains of prime ideals do have same length

A noetherian local domain, $\dim(A) = 1$

$(0) \subsetneq m_A$

Only prime ideals are (0), m_A

A DVR $\Leftrightarrow$ A integrally closed in its fraction field

$A = \{a_0 + a_2 x^2 + a_3 x^3 + \cdots\} \subset F[[x]]$

$m_A = (x^2, x^3)$, not DVR

Lecture 14 (2009-03-17)

3/17/09

Dedekind domains

1) 2) 3)

Def. A ring, A Dedekind domain $\iff$ A noeth, $\dim(A)=1$, and A integrally closed

Thm. Equiv to: fractionary ideals form a group

Let A integral domain, $K =$ quotient field of A

J is a fractionary ideal of A if J is an A-module under $a \cdot x = ax \in K$, and
$\exists\, b \neq 0 \in A : bJ \subseteq A$ ("bounded denominators")

The fractionary ideals form comm, assoc, semigroup with id under $(I, J) \mapsto IJ$
$= \sum a_i b_i$, for $a_i \in I$, $b_i \in J$

identity $= A$

Prop. This semigroup is a group $\iff$
every fractional ideal is invertible $\iff$
every ideal is invertible

J fractional, then J invertible $\iff$ bJ for any $b \neq 0$ (because $(b)^{-1} = (b^{-1})$)

Def. A is a DVR if A is a local Dedekind domain.

Facts. A local domain, noeth, 1-dim, $m = m_A$
1) $\overset{\text{This}}{\Longrightarrow}$ I ideal of A, $\neq (0)$, $\exists n : m^n \subseteq I$
2) $m^n \neq m^{n+1}$ $\forall n$

Thm. TFAE for A local domain, noeth, 1-dim. (write $m = m_A$)
- * 1) A is a DVR (old style) A is valuation ring
- * 2) A is integrally closed
- 3) m is principal
- * 4) $\dim_{A/m} m/m^2 = 1$
- 5) every non-zero ideal is a power of m
- 6) $\exists\, x \in A$: every ideal $\neq 0$ is (x^n)

Pf. $1 \Rightarrow 2$ we did for any valuation ring

$2 \Rightarrow 3$: let $a \in m$, $a \neq 0$: $\exists n : m^n \subseteq (a)$, $m^{n-1} \not\subseteq (a)$.

let $b \in m^{n-1}$, $b \notin (a)$. Let $x = a/b$ (b can't be 0).

Then $x^{-1} \notin A$ $[x^{-1} \in A \Rightarrow b/a \in A \Rightarrow b = b/a \cdot a \in (a)]$.

$\Rightarrow x^{-1}$ not integral over A

$\Rightarrow x^{-1}m \not\subseteq m$ (by integrality test)

(m is a faithful $A[x^{-1}]$-module, f.g. as an A-module.

$x^{-1}m \subseteq A$, (because its $\frac{b}{a}m$, and $\frac{m^n}{a} \in A$, $b \in m^{n-1}$, and $\Rightarrow x^{-1}m = A$

because if $x^{-1}m \not\subseteq m$, $x^{-1}m = A$, and so $m = (x)$.

$3 \Rightarrow 4$. clear; $m/m^2 = m \otimes_A A/m$, which is generated by $\bar{x}$ as an A/m-module.

$M \otimes_A A/I \cong M/IM$, and $I \to A \to A/I \to 0$

$I \otimes M \xrightarrow{\phi} M \to (A/I) \otimes M \to 0$

image $\phi = IM$

$4 \Rightarrow 5$ $I \subseteq m$,

$\exists r : I \subseteq m^r$, but $I \not\subseteq m^{r+1}$

$m^s \subseteq I$, $m^s \not\subseteq m^{s+1} \Rightarrow I \not\subseteq m^{s+1}$

$m = (x)$, $\exists y \in I$: $y = ax^r$, but $y \notin (x^{r+1})$

$a \in A$

$\Rightarrow a \notin (x) \Rightarrow a \notin m \Rightarrow a$ is a unit in A

$(y) = (x^r)$, $m^r = (y) \subseteq I \subseteq m^r$

$5 \Rightarrow 6$. $\exists x \in m : x \notin m^2$

$(x) = m^r \Rightarrow r = 1$, $(x) = m$

$m^r = (x^r)$

$6 \Rightarrow 1$ $m = (x)$

$(x^k) \neq (x^{k+1})$

$a \neq 0 \in A$, $\exists k : a \in (x^k)$, $a \notin (x^{k+1})$

$v(a) = k$ Check this is a valuation

if $x = a/b \in K$, $v(x) = v(a) - v(b)$

3/17/09

Let A be integral domain, K = quotient field of A

$I \subseteq A$, $I \neq 0$ is invertible $\Leftrightarrow$ $\exists a_1, \dots, a_n \in I$, $q_1, \dots, q_n \in K$:

1) $q_i I \subseteq A$

2) $1 = \sum_{i=1}^{n} a_i q_i$

If this is true, let $J = \langle q_1, \dots, q_n \rangle$

$JI \subseteq A$, $1 \in JI$, so $JI = A$

$q_i = b_i/c_i$, $c = \prod c_i$ $cJ \subseteq A$

. J fractional ideal

$IJ = A \Rightarrow 1 = \sum_{i=1}^{n} a_i q_i$ $a_i \in A$, $q_i \in J$

$q_i I \subseteq IJ = A$

Thm. I invertible $\Leftrightarrow$ I projective

Lemma. Let M be an A-module

M is projective $\Leftrightarrow$ $\exists$ family of elements $x_\alpha \in M$, maps $\phi_\alpha : M \to A$

such that $x \in M \Rightarrow x = \sum_\alpha \phi_\alpha(x) x_\alpha$ ($\phi_\alpha(x) = 0$ almost all α)

Pf. ($\Leftarrow$) $\alpha \in I$

$A^I \xrightarrow{\pi} M$ $=$ $A^I \simeq \phi(M) \oplus \mathrm{Ker}(\pi)$

$\varepsilon_\alpha \mapsto x_\alpha$

ϕ_α determine $\phi : M \to A^I = \overset{\oplus}{\coprod_I} A \subseteq \prod_I A$

$\phi(m) = \{\phi_\alpha(m)\}$

$\pi \phi = id_M$

$\pi\phi(x) = \pi\{\phi_\alpha(x)\} = \sum_\alpha \phi_\alpha(x) x_\alpha = x$

($\Rightarrow$) M projective; if x_α generate M, $I = \{\alpha\}$

Let $\phi_\alpha = p_\alpha \circ f$

$\pi f = id$

$x = \sum \phi_\alpha(x) x_\alpha$

$$\begin{tikzcd} A^I \arrow[r, two heads, "\pi"] & M \\ & M \arrow[u, "id"'] \arrow[ul, dashed] \end{tikzcd}$$

$\varepsilon_\alpha \mapsto x_\alpha$

M projective $\Rightarrow$ lift identity to $M \xrightarrow{f} A^I \xrightarrow{p_\alpha} A$

$\prod\{\phi_\alpha(x)\} = f(x) = \sum \phi_\alpha(x) x_\alpha = \pi(f(x)) = y$

$q_1, q_2, \dots, q_n \qquad x_i = a_i$

M invertible, $\phi_\alpha : M \to A \quad \phi_\alpha(x) = \underset{\in A}{q_\alpha} x \quad x_\alpha = a_\alpha$

$$\sum \phi_\alpha(x) x_\alpha = \sum q_\alpha x a_\alpha = \sum (q_\alpha a_\alpha) x = 1 \cdot x = x$$

projective $\Rightarrow$ invertible

$\phi_\alpha : I \to A \qquad x \in I \Rightarrow x = \sum_\alpha \phi_\alpha(x) x_\alpha$ identities in K

$x_\alpha \in I$

$\phi_\alpha(xy) = x\phi_\alpha(y) = y\phi_\alpha(x) \qquad x, y \neq 0$

so $\dfrac{\phi_\alpha(y)}{y} = \dfrac{\phi_\alpha(x)}{x}$ let $q_\alpha = \dfrac{\phi_\alpha(x)}{x}$, indep. of x

$q_\alpha = 0$ for all but finitely many α

$x = \sum_\alpha \phi_\alpha(x) x_\alpha = \sum_\alpha x q_\alpha x_\alpha =$

$x\left(\sum_\alpha q_\alpha x_\alpha\right)$

$1 = \sum_\alpha q_\alpha x_\alpha \qquad q_\alpha x = \phi_\alpha(x) \in A$ any $x \in I$

$q_\alpha I \subseteq A$

Corollary: invertible $\Rightarrow$ finitely generated

$x = \sum_\alpha q_\alpha x x_\alpha = \sum \underset{\in A}{(q_\alpha x)} x_\alpha \in I$

Thm (next time) A noeth, M f.g. A-module

M projective $\Leftrightarrow$ M locally free

M_p free A_p-module $\forall$ prime p

Lecture 15 (2009-03-19)

3/19/09

A Dedekind domain iff

A noeth, integrally closed, 1-dimensional, domain

(recall 1-dim if all non-zero prime ideal is maximal)

If A domain, then A Dedekind domain $\Leftrightarrow$ A noeth, and A_p is a DVR $\forall$ non-zero prime ideals p

If A domain, then A integrally closed $\Leftrightarrow A_p$ integrally closed $\forall$ prime ideals p

$\Leftrightarrow A_m$ int closed $\forall$ max ideals m.

Let $A \subseteq B$ rings, let C = int closure of A in B, S = mult. closed subset of A

then C_S = int closure of A_S in B_S

Pf. $c/s \in C_S$, $c \in C$, $s \in S$

then $c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 = 0$, $a_i \in A$, thus

$$c^n/s^n + \frac{a_{n-1}}{s}\frac{c^{n-1}}{s^{n-1}} + \cdots \frac{a_1}{s^{n-1}}\frac{c}{s} + \frac{a_0}{s^n} = 0$$, so C_S is integral over A_S

Let $x \in B_S$ be integral over A_S.

$$x^n + \frac{a_{n-1}}{s_{n-1}}x^{n-1} + \cdots \frac{a_1}{s_1}x + \frac{a_0}{s_0} = 0$$

Let $s = s_{n-1} \cdots s_1 s_0$

$$s^n x^n + \frac{s^n a_{n-1}}{s_{n-1}}x^{n-1} + \cdots \frac{s^n a_1 x}{s_1} + \frac{s^n a_0}{s_0} = 0$$

$$(sx)^n + \frac{s a_{n-1}}{s_{n-1}}(sx)^{n-1} + \cdots \frac{s^{n-1}a_1}{s_1}(sx) + \frac{s^n a_0}{s_0} = 0$$

so sx integral $/A$

$sx \in C$, $x = \frac{sx}{s} \in C_S$

Now take $A \to C$, A domain, B = quotient field of A. $A \to C$ is an iso iff
C = int closure of A in B,
A integrally closed, and $A \xrightarrow{\sim} C$ iso $\Leftrightarrow A_p \xrightarrow{\sim} C_p$ $\forall p$ (in general, true of modules; $M=0$ iff $M_p = 0$ $\forall p$, then apply to ker + coker), and $A_p \xrightarrow{\sim} C_p$ iso $\forall p$ iff A_p int closed $\forall P$. Thus C_p = int closure of A in $B_p = B$ (B is a field)

1-dimensional is taken care of, because if we had $0 \subsetneq P \subsetneq Q$, then

$0 \subseteq PA_Q \subsetneq QA_Q$

① ②

Thm. A Dedekind domain $\Leftrightarrow$ fractional ideals form group $\Leftrightarrow$.
(last time)
every fractional ideal is invertible $\Leftrightarrow$
every ideal is invertible $\Leftrightarrow$
every ideal is projective

For $2 \Rightarrow 1$, we know invertible $\Rightarrow$ finitely generated

$2 \Rightarrow A$ noeth ; every ideal invertible $\Rightarrow$ every ideal of A_p is invertible $\Rightarrow$ every ideal of A_p projective, but A_p local, so projective ideals are free, and for any domain, free $\Rightarrow$ principal, so every A_p is PID and A noeth, so A_p DVR $\forall p$

to get $1 \Rightarrow 2$, A noeth, M f.g. A-module, M projective $\Leftrightarrow M_p$ is free $\forall p$ prime,

a) $\mathrm{Hom}_A(M,N)_p \xrightarrow{\sim} \mathrm{Hom}_{A_p}(M_p, N_p)$

PROJECTIVE
$\Leftrightarrow$
LOCALLY FREE

Pf. True for $M=A$, true for $M=A^n$.

M f.g., A noeth $\Rightarrow M$ finitely <u>presented</u> (relations defining $A^n \to M$ also f.g)

So get $A^m \to A^n \to M \to 0$ $\qquad A^m_p \to A^n_p \to M_p \to 0$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M,N)_p \arrow[r] \arrow[d] & \mathrm{Hom}_A(A^n,N)_p \arrow[r] & \mathrm{Hom}_A(A^m,N)_p \\
0 \arrow[r] & \mathrm{Hom}_{A_p}(M_p,N_p) \arrow[r] & \mathrm{Hom}_{A_p}(A^n_p,N_p) \arrow[r] & \mathrm{Hom}_{A_p}(A^m_p,N_p)
\end{tikzcd}$$

Want to show I projective; equiv to $\mathrm{Ext}^1_A(I,N)=0 \ \forall N$ (i.e, Homming out of I is exact). So we have

$$A^m \to A^n \to I \to 0$$

$$0 \to K \to A^n \to I \to 0 \quad , \quad 0 \to K_p \to A^n_p \to I_p \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(I,N)_p \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(A^n,N)_p \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(K,N)_p \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^1_A(I,N)_p \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{A_p}(I_p,N_p) \arrow[r] & \mathrm{Hom}_{A_p}(A^n_p,N_p) \arrow[r] & \mathrm{Hom}_{A_p}(K_p,N_p) \arrow[r] & \mathrm{Ext}^1_{A_p}(I_p,N_p) \arrow[r] & 0
\end{tikzcd}$$

← we get a map here and know it is an iso

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But $\mathrm{Ext}^1_{A_P}(I_P, N_P) = 0$ $\forall P$ because assuming Dedekind domain, which we showed meant that every ideal is locally free, so all $\mathrm{Ext}^1_{A_P}(I_P, N_P) = 0$. Since $\mathrm{Ext}_A^1(I, N)_P \simeq \mathrm{Ext}^1_{A_P}(I_P, N_P)$, $\mathrm{Ext}^1_A(I, N)_P = 0$ $\forall P$, so $\mathrm{Ext}_A^1(I, N) = 0$, so I projective.

Thm. A Dedekind domain, then every non zero ideal is the product of prime ideals, unique up to order, and every fractionary ideal is the product of prime ideals and inverses of prime ideals uniquely up to order.

Equivalently, the ideal gp is the free ab gp on the prime ideals.

Pf. Let $I \neq 0$ be ideal. $I \subseteq M_1$ maximal; if $I = M_1$, stop.

lemma, $J^{-1} = (A : J) = \{x \in K : xJ \subseteq A\}$

$IJ = A$,
$I = J^{-1}$

$I \subseteq (A:J) = (A:J)JI \subseteq AI = I$

$I \subseteq J \Rightarrow J^{-1} \subseteq I^{-1}$

So $I \subsetneq IM_1^{-1} \subsetneq A$

$M_1 \subsetneq A \Rightarrow A^{-1}$
$\quad \|$
$A \subsetneq M_1^{-1}$

$I \subsetneq IM_1^{-1}$

So $IM_1^{-1} \subseteq M_2$ max; if $=$, we're done since $I = M_1 M_2$, if not, keep going, we get

$I \subsetneq IM_1^{-1} \subsetneq IM_1^{-1}M_2^{-1} \cdots$ etc ascending chain; noeth means it must stop, so $I = M_1 M_2, \ldots M_n$.

$P_1 \supseteq P_1 \cdots P_n = Q_1 Q_2 \cdots Q_m$

ordinary ideals

$IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$

$I_1 \cdots I_n \subseteq P \Rightarrow$ some $I_j \subseteq P$

By renumbering, $Q_1 = P_1$, so $P_2 \cdots P_n = Q_2 \cdots Q_m$

J a fractionary ideals then $bJ = I$ for some $b \neq 0 \in A$

$bJ = (b)J$, and $(b) = Q_1 \cdots Q_n$ prime, $I = P_1 \cdots P_m$ primes,

so $J = P_1 \cdots P_n Q_1^{-1} \cdots Q_n^{-1}$.

$\overset{C}{\text{Ideal gp}} \supseteq \overset{B}{\{\text{principal ideals}\}}$; $C/B \overset{\text{def}}{=}$ class group of Dedekind domain

Claim. $[F : \mathbb{Q}] < \infty$, $O_F =$ int closure of $\mathbb{Z}$ in F ; O_F Dedekind domain

Pf. 1) O_F integrally closed ✓

Thm. Let A be integrally closed in its quotient field K

Let L be a a finite <u>separable</u> extension

Let $B =$ integral closure of A in L

Then $\exists v_1, \ldots, v_n \in L$: $B \subseteq$ sub A-module of L generated by $v_1, \ldots, v_n$

Pf. Take basis for L/K, $x_1, \ldots, x_n$.

Let $x = x_i$

$$y^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0 \quad , c_i \in K, \text{ so } c_i = a_i/b_i$$

for $a_i, b_i \in A$

$$x^n + \frac{a_{n-1}}{b_{n-1}} x^{n-1} + \cdots + \frac{a_1}{b_1} x + \frac{a_0}{b_0} = 0$$

$$b = b_{n-1} \cdots b_1 b_0 \in A$$

$$(bx)^n + \frac{a_{n-1} b}{b_{n-1}} (bx)^{n-1} + \cdots \frac{a_1}{b_1} b^{n-1}(bx) + \frac{a_0 b^n}{b_0} = 0$$

so bx integral over A, so $bx \in B$. Thus $\exists b \in A$: $bx_1, bx_2, \ldots bx_n \in B$,
this remains a basis for L/K,

So $\exists$ basis $v_1, \ldots, v_n$ of L/K, $v_i \in B$.

Look at bilinear form $(x,y) \mapsto Tr_{L/K}(x,y)$ on L. Non-degenerate, since
$Tr(x,y) = 0 \; \forall y \Rightarrow x = 0$, and $\Leftrightarrow Tr \neq 0$

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Choose dual basis $v_1, \ldots v_n$, i.e $Tr(u_i v_j) = \delta_{ij}$

$x \in B$, $x = \sum_{j=1}^{n} x_j v_j$, $x_j \in K$

$$Tr(xu_i) = \sum_{j=1}^{n} Tr(x_j v_j u_i) = \sum_{j=1}^{n} x_j Tr(v_j u_i) = \sum_{j=1}^{n} x_j \delta_{ij} = x_i$$

so $xv_i \in B$, so $Tr(xu_i) \in A$, so $x_i \in A$ because

$x \in$
$$\begin{tikzcd} L \arrow[d, dash] \\ K[x] \arrow[d, dash] \\ K \end{tikzcd}$$

$$x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0 = 0$$

$$Tr_{K[x]/K}(x) = -a_{n-1}$$

we have been
conflating A noeth
$\sum A v_j$ f.g./A noeth
B noeth

Thus 2) O_F is noeth

Lecture 16 (2009-03-31)

Lemma 1. $\overset{A \subseteq B}{B}$ integral over A, A, B int domains A field $\Leftrightarrow$ B field 3/31/09

Pf. A is a field; for $b \in B$, $b \neq 0$ $\quad b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$, $a_i \in A$

we don't know we can divide by b yet, but this is just to set up

So $a_0/b = -b^{n-1} - a_{n-1}b^{n-2} - \cdots - a_1$

so $1/b = \frac{1}{a_0}(\qquad)$

in A

in A

So $\frac{1}{b} \in B$, so B field

B is a field; $a \in A$, $a \neq 0$, $\frac{1}{a} \in B$, so $\frac{1}{a}$ integral over A,

so $\left(\frac{1}{a}\right)^n + a_{n-1}\left(\frac{1}{a}\right)^{n-1} \quad + a_0 = 0$, so $\frac{1}{a} + a_{n-1} + a_{n-2}a \cdots + a_0 a^{n-1} = 0$

So $\frac{1}{a} \in A$

Lemma 2. $\overset{A \subseteq B}{B}$ integral over A, Q prime $\subseteq B$.

Q maximal $\Leftrightarrow$ $Q \cap A$ maximal

Pf. Apply Lemma 1 to $A/Q \cap A \hookrightarrow B/Q$

Lemma 3. $\overset{A \subseteq B}{B}$ integral over A, Q, Q' prime ideals of B, $Q \subset Q'$. Then

$Q \cap A = Q' \cap A \Rightarrow Q = Q'$

Let $P = Q \cap A = Q' \cap A$; P prime $\subseteq A$. Let M extension of P to A_P, N, N' extension of Q, Q' to $B_P = B_{A-P}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A_P \arrow[r] & B_P
\end{tikzcd}$$

$N \subseteq N'$, $N \cap A_P = M = N' \cap A_P$, and M max $\Rightarrow$

N, N' max $\Rightarrow$ $N = N' \Rightarrow Q = Q'$.

Thm. $A \subseteq B$, B integral over A, P prime ideal $\subseteq A$ $\Rightarrow$ $\exists$ Q prime ideal $\subseteq B$: $P = Q \cap A$. In other words, the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective

Pf. Let N be max ideal of B_P. $N \cap A_P$ is a max ideal of A_P, $N \cap A_P = PA_P$

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
A_P \arrow[r, hook] & B_P
\end{tikzcd}$$

localization is exact, which is why this is injective

Let $Q = \beta^{-1}(N)$ $\quad Q \cap A = P$

$\alpha^{-1}(N \cap A_P) = P$

Going up thm

$A \subseteq B$, B int$/A$

If $P_1 \subseteq P_2 \subseteq \quad \subseteq P_n$ chain of prime ideals in A

$Q_1 \subseteq Q_2 \subseteq \quad \subseteq Q_m$ chain of prime ideals in B, $m \leq n$

with $Q_i \cap A = P_i$, $1 \leq i \leq m$.

Then $\exists\ Q_{m+1} \subseteq Q_{m+2} \subseteq \cdots \subseteq Q_n$ extending Q_m

with $Q_i \cap A = P_i$, $m \leq i \leq n$.

Pf. Reduce to case $m=1$, $n=2$,

$A \subseteq B$, B int$/A$, Q_1 prime $\subseteq B$, $Q_1 \cap A = P_1$

$P_1 \subseteq P_2$

Let $\overline{A} = A/P_1$, $\overline{B} = B/Q_1$, so $\overline{A} \subseteq \overline{B}$ $\overline{B}$ int$/\overline{A}$

$\exists$ prime ideal $\overline{Q_2}$ of $\overline{B}$ lying over $\overline{P_2}$

$$\begin{tikzcd} P_2 & Q_2 \\ A \arrow[r] \arrow[d] & B \arrow[d] \\ \overline{A} \arrow[r] & \overline{B} \\ \overline{P_2} & \overline{Q_2} \end{tikzcd}$$

Kähler differentials

($\Omega_{B/A}$) Let $A \to B$, Let M be a B-module; An A-derivation of B into M is a map $D: B \to M$ with

1) $D(x+y) = D(x) + D(y)$
2) $D(xy) = xD(y) + yD(x)$
3) $a \in A \Rightarrow D(a) = 0$

Equivalently, require that $D(ax) = aD(x)$

because then $D(a1) = aD(1) + D(a)$

$\{A\text{-derivations of } B \text{ to } M\} = Der_A(B, M)$, this is a B-module

check 2) for $D_1 + D_2$ $\quad$ $b \in B$, $D \in Der_A(B, M)$

$(bD)(x) = bD(x)$

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$\Omega_{B/A}$ is a B-module together with an A-derivation $d: B \to \Omega_{B/A}$ such that for any B module M and A-derivation $D: B \to M$, $\exists$ unique $\phi \in \mathrm{Hom}_B(\Omega_{B/A}, M)$, $D = \phi \circ d$.

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega_{B/A} \arrow[d, "\phi"] \\ & M \end{tikzcd}$$

To define $\Omega_{B/A}$, take free B-module F on $\langle db \rangle$, $b \in B$. Divide by the B-module R generated by

1) $\{\langle d(b_1+b_2)\rangle - \langle db_1\rangle - \langle db_2\rangle\}$
2) $\{\langle d(b_1 b_2)\rangle - b_1\langle db_2\rangle - b_2\langle db_1\rangle\}$
3) $\langle d\, f(a)\rangle$

$\Omega_{B/A} : F/R$

$\omega \mapsto \overline{\omega}$

$F \to F/R$

works by construction

Given $D: B \to M$, define ϕ by $\phi(\langle db\rangle) = D(b)$

$F \to M$

this vanishes on R, because D is a derivation.

We can also construct $\Omega_{B/A}$ as follows:

$B \underset{A}{\otimes} B \xrightarrow{\psi} B$

$b_1 \otimes b_2 \mapsto b_1 b_2$

let $I = \mathrm{Ker}(\psi)$

Then $\Omega_{B/A} = I/I^2$

What is d?

$M/IM = M \underset{C}{\otimes} B$

equivalently, $I/I^2 = I \underset{B \underset{A}{\otimes} B}{\otimes} B$

M a C-module $B = C/I$

If
$$\begin{tikzcd} B \arrow[r] & C \\ A \arrow[u] \arrow[ur] \arrow[ur, phantom, "G"] & \end{tikzcd}$$
, $\exists$ exact sequence

$$\Omega_{B/A} \underset{B}{\otimes} C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

This is exact $\Leftrightarrow$ $\forall$ C-modules M,

$$0 \to \mathrm{Hom}_C(\Omega_{C/B}, M) \to \mathrm{Hom}_C(\Omega_{C/A}, M) \to \mathrm{Hom}_C(\Omega_{B/A} \underset{B}{\otimes} C, M)$$

exact

$\mathrm{Hom}_B(\Omega_{B/A}, M)$

$\|$

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$ $\leftarrow$ "trivial"

Put
$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & B \arrow[r, two heads] & C \arrow[r] & 0 \\ & & A \arrow[u] \arrow[ur] & & \end{tikzcd}$$

M is a C-module

$\Omega_{C/B} \to 0$

$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$ is exact

$0 \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \xrightarrow{\phi} \mathrm{Hom}_B(I, M)$

$\mathrm{Hom}_C(I/I^2, M) = \mathrm{Hom}_C(I \otimes_B C, M) = \mathrm{Hom}_B(I, M)$

Define $\phi(D) = D|_I$ $\quad x \in I,\ y \in B,\ D(yx) = yD(x) + \underset{0}{\underset{\shortparallel}{xD(y)}}$

so $D(yx) = yD(x)$,

so is a B-homomorphism

Example: $B = A[x_1, \dots, x_n]$

$\Omega_{B/A} \simeq$ free B-mod on $dX_1, dX_2, \dots, dX_n$

If M B-module,

$\mathrm{Hom}_B(F_n, M) \quad dX_i \mapsto \underset{y_i \in M}{\text{any}}$

also, $D(p(x_1, \dots, x_n)) = \sum \frac{\partial p}{\partial x_i}(x_1, \dots, x_n) D(x_i)$

Example: K/k finite separable $\Rightarrow \Omega_{K/k} = 0$

$K = k[\alpha] = k[x]/(f)$ for f separable $(f'(\alpha) \neq 0)$

To show $\Omega_{K/k} = 0$, enough to show $\mathrm{Hom}_K(\Omega_{K/k}, K) = 0$

$\mathrm{Der}_k(K, K)$

Let $D(x) = b$

$D(g(x)) = g'(x) D(x)$

$D(f(x)) = f'(x) D(x)$

$0 = D(f(\alpha)) = f'(\alpha) D(\alpha)$

$f'(\alpha) \neq 0$ so $D(\alpha) = 0$

$$\begin{tikzcd} k[x] \arrow[r] & K \\ k \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

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Thm. Let $K = k(y_1, \dots, x_n)$ be an f.g. field extension. Then $\Omega_{K/k}$ is a finite dim K-vector space and $\dim \Omega_{K/k} \geq$ transcendence degree K/k with equality $\iff$ K/k is separably generated.

Recall $K = k(x_1, \dots, x_n)$, then transcendence deg of $K/k = r$ if $\exists\, y_1, \dots, y_r \in K$: $y_1, \dots, y_r$ are alg. indep. and K is alg over $k(y_1, \dots, y_r)$. (Need to prove independent $y_1, \dots, y_r$)

transcendence basis

K/k is separably generated when $\exists\, y_1, \dots y_r$ as above, with $K/k(y_1, \dots, y_r)$ is <u>separable</u> + algebraic.

Note: not all transcendence bases tell separable:

$$\begin{tikzcd} k(x) \arrow[d, dash] \\ k \end{tikzcd}$$

x separable basis

x^p not

Lecture 17 (2009-04-02)

4/2/09

$A \to B$, $S \subseteq B$ mult. closed

$\Omega_{B_S/A} = (\Omega_{B/A})_S$

$$\begin{tikzcd} B \arrow[r] & B_S \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

tensoring with B_S = localizing →

$$\begin{tikzcd} \Omega_{B/A} \otimes_B B_S \arrow[r, "\phi"] \arrow[d, equal] & \Omega_{B_S/A} \arrow[r] & \Omega_{B_S/B} \arrow[r] & 0 \\ (\Omega_{B/A})_S & & & \end{tikzcd}$$

So our statement is that ϕ is an isomorphism

For M an B_S-mod,

$$\begin{tikzcd} \mathrm{Hom}_{B_S}(\Omega_{B_S/A}, M) \arrow[r, phantom, "="] \arrow[d, equal, "?"] & \mathrm{Der}_A(B_S, M) \arrow[dd, "\text{Defined by}"] & \tilde{D}(b/s) \\ \mathrm{Hom}_{B_S}(\Omega_{B/A} \otimes B_S, M) & & = \frac{sD(b) - bD(s)}{s^2} \arrow[d] \\ \mathrm{Hom}_B(\Omega_{B/A}, M) \arrow[r, phantom, "="] & \mathrm{Der}_A(B, M) \arrow[r, phantom, "\ni"] & D \end{tikzcd}$$

If we can extend D to $\tilde{D}$, $\tilde{D}(b) = D(b) =$

$D(b/s \cdot s) = s\tilde{D}(b/s) + b/s\, D(s)$

$s\tilde{D}(b/s) = D(b) - b/s\, D(s)$

$\tilde{D}(b/s) = \frac{sD(b) - bD(s)}{s^2}$

Check that doesn't depend on how you write b/s, that $\tilde{D}$ is a derivation, and check that this formula is forced

Given
$$\begin{tikzcd} B & \\ A \arrow[u] \arrow[r] & A' \end{tikzcd}$$
, let $B' = B \otimes_A A'$, so
$$\begin{tikzcd} B \arrow[r, dashed] & B' \arrow[r, "D"] & M \\ A \arrow[u] \arrow[r] & A' \arrow[u, dashed] & \end{tikzcd}$$
M a B'-module

Then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$

$$\begin{tikzcd} B \arrow[r, "D"] \arrow[d] & M \\ B' \arrow[ur, dashed, "\tilde{D}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}_{B'}(\Omega_{B'/A'}, M) \arrow[r, "\sim"] & \mathrm{Hom}_{B'}(\Omega_{B/A} \otimes_B B', M) \arrow[d, phantom, "\wr"] \\ & \mathrm{Hom}_B(\Omega_{B/A}, M) \arrow[d, equal] \\ \mathrm{Der}_{A'}(B', M) \arrow[r, "\sim"] & \mathrm{Der}_A(B, N) \end{tikzcd}$$

$B \otimes_A A' \to M$ $\quad$ $B \times A' \to M$ $\quad$ $\tilde{D}(b, a') = a' D(b)$

A-bilinear

$\tilde{D}(xy) = x\tilde{D}(y) + y\tilde{D}(x)$

$x = b_1 \otimes a_1'$, $y = b_2 \otimes a_2'$

$$\hat{D}\big((b_1\otimes a_1')(b_2\otimes a_2')\big) \overset{?}{=} (b_1\otimes a_1')\tilde{D}(b_2\otimes a_2') + (b_2\otimes a_2')\tilde{D}(b_1\otimes a_1')$$

$\|$

$\tilde{D}(b_1b_2 \otimes a_1'a_2')$ $\qquad\qquad$ $(b_1\otimes a_1')a_2'D(b_2) + (b_2\otimes a_2')a_1'D(b_1)$

$\|$

$a_1'a_2'\tilde{D}(b_1b_2)$

$(b\otimes a)m = bam$

Thm. Let K be f.g. field extension of k. Then $\dim_K(\Omega_{K/k}) =$ tr. deg. $K/k \iff K$ is sep. generated $/k$.

Lemma. Let K/k be finite, separable.

Let L be a field $\supseteq K$

Let D be a derivation of $k \to L$

Then $\exists!$ extension of D to a derivation of K into L

We know $K = k[x]/(f)$, f separable irreducible

Given $D: k \to L$ derivation

extend to $k[x]$ (take any $\beta \in L$, define $D(x) = \beta$)

$g(x) = b_m x^m + \cdots + b_0$

$$D(g(x)) = mb_m x^{m-1}D(x) + (m-1)b_{m-1}x^{m-2}D(x) \cdots + \underbrace{b_1D(x) + D(b_m)x^m + D(b_{m-1})x^{m-1} + \cdots + D(b_0)}_{\text{call this } dg}$$

Check that this is a derivation of $k[x]$ to L

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want $\tilde{D}$ to be defined on $K = k[x]/f$, so

$$\tilde{D}(f(x)) = 0$$

$$f'(x)\beta + df = 0$$

$$\downarrow$$

$$f'(\alpha)\beta + df = 0$$

$$\beta = -df/f'(\alpha) \qquad f'(\alpha) \neq 0 \text{ since separable}$$

we want any multiple of f to go to 0 too, but D isn't a multiplicative homomorphism,

so we have to check $\tilde{D}(hf) = 0$

$$\tilde{D}(hf) = f\tilde{D}(h) + h\tilde{D}(f)$$

$$f\tilde{D}(h) + h \cdot \overset{\shortparallel}{0}$$

$$\downarrow$$

$$f(\alpha)\tilde{D}(h) = 0$$

Cor.

$$\begin{tikzcd} K \arrow[r, "\text{finite}", "\text{sep}"'] & L \\ k \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$\Omega_{K/k} \otimes_K L \xrightarrow{\sim} \Omega_{L/k} \qquad (\rightarrow \Omega_{L/K} = 0)$$

both L-vector spaces; suffices to prove dual spaces are $\simeq$

$$\begin{tikzcd} \mathrm{HOM}(\Omega_{K/k}, L) \arrow[r] \arrow[d, equal] & \mathrm{Der}_k(L,L) \arrow[dl] \\ \mathrm{Der}_k(K,L) & \end{tikzcd}$$

↑ but we proved ∃! extension, so this is iso ~

($\Leftarrow$) L is sep generated $\exists K = k(x_1, \ldots, x_r)$, L/K is finite sep

$x_1, \ldots, x_r$ alg independent

$$\begin{tikzcd} K \arrow[r] & L \\ k \arrow[u] \arrow[ur] & \end{tikzcd} \qquad \Omega_{L/k} \simeq \Omega_{K/k} \otimes_K L$$

$$\Omega_{\overset{A}{\overset{\shortparallel}{k[x_1, \ldots, x_r]}}/k} = \text{free module in } dx_1, \ldots, dx_r$$

$$k(x_1, \ldots, x_r) = k[x_1, \ldots, x_r]_{A - \{0\}} \qquad \Omega_{k(x_1, \ldots, x_r)/k} = \text{free } k(x_1, \ldots, x_r)\text{-module on } dx_1, \ldots, dr$$

this free K-mod on $dx_1 \cdots dx_r$ K-vector space of dim r ✓

Lemma. If L/K f.g., $\Omega_{L/K} = 0 \Rightarrow L/K$ finite sep

Pf. Take tr. basis of L/K

$$\begin{tikzcd} K(x_1, \ldots, x_r) \arrow[r, dash, "\text{finite alg}"] & L \\ K \arrow[u, dash] \arrow[ur, dash, bend right] & \end{tikzcd}$$

If $L/K(x_1, \ldots, x_r)$ is sep

$$\Omega_{L/K} = \Omega_{K(x_1, \ldots, x_r)/K} \otimes_K L$$

$r > 0 \Rightarrow \Omega_{L/K} \neq 0$ $r = 0$

If $L/K(x_1, x_2, \ldots x_r)$ is not separable

ERASED adjoin everything except the p^{th} root of something

let that be L'' D must be 0 on L''

So $\exists$ non trivial derivation $L \to L$, so certainly $K \to L$?!?

Suppose $\Omega_{K/k}$ has dim $=$ tr deg K/k $K = k(x_1, \ldots, x_n)$

$\Omega_{K/k}$ is generated by $dx_1, \ldots dx_n$

By renumbering, assume $dx_1, \ldots, dx_r$ basis, so $r =$ tr. deg K/k

let $K' = k(x_1, \ldots, x_r)$

$$\begin{tikzcd} K' \arrow[r] & K \\ k \arrow[u, dash] \arrow[ur] & \end{tikzcd} \quad \Omega_{K'/k} \otimes_K K' \xrightarrow{\phi} \Omega_{K/k} \to \Omega_{K/K'} \to 0$$

ϕ is surjective $\Rightarrow \Omega_{K/K'} = 0 \Rightarrow K/K'$ is finite separable $\Rightarrow$

tr deg $K'/k =$ tr deg $K/k = r$

K is sep generated $/k$

$K' = k(x_1, \ldots, x_r) \Rightarrow x_1, \ldots, x_r$ are alg independent

There is an object $N_{L/K}$ 4/2/09

$$0 \to N_{K/k} \otimes_K L \to N_{L/K} \to N_{L/k} \to \Omega_{K/k} \otimes_K L \to \Omega_{L/K} \to \Omega_{L/K} \to 0$$

tr deg $= \dim \Omega_{L/K} - \dim N_{L/K}$

$N_{L/K} = 0$ iff finite sep

A_α ab gps (or R-modules, or rings, etc)

I directed set (poset, $x, y \in I \Rightarrow \exists z \in I : z \geq x, z \geq y$)

$\beta \geq \alpha$ $\phi_{\beta\alpha} : A_\beta \to A_\alpha$ (coherent)

$$\begin{tikzcd} A_\gamma \arrow[r, "\phi_{\gamma\beta}"] \arrow[rr, bend right, "\phi_{\gamma\alpha}"'] & A_\beta \arrow[r, "\phi_{\beta\alpha}"] & A_\alpha \end{tikzcd}$$

$$\varprojlim A_\alpha \subseteq \prod_\alpha A_\alpha$$

$$\{(a_\alpha) \in \prod_\alpha A_\alpha : \phi_{\beta\alpha}(a_\beta) = a_\alpha\}$$

Lecture 18 (2009-04-07)

4/7/09

I directed set, M_i for $i \in I$

$i \geq j$, $\phi_{ij} : M_i \to M_j$ $\quad \forall M_k$

$$\begin{tikzcd} M_i \arrow[r] \arrow[dr, "a"'] & M_j \arrow[d] \\ & M_k \end{tikzcd}$$

$\phi_{ii} = id$

$\varprojlim_{i \in I} M_i \subseteq \prod_i M_i$, with $= \{\{x_i\} : \phi_{ij}(x_i) = x_j\}$

If M_i are top. grps, put product topology on $\prod M_i$

$\varprojlim$ is a closed subset of the product

If the M_i are compact (every cover has finite subcover), Tychonoff product thm $\Rightarrow \prod M_i$ compact, but only closed subsets of compact Hausdorff are compact, so $\varprojlim$ need not be compact.

Direct limit is not inside product, so doesn't have natural topology

If M_i are finite, discrete $\Rightarrow \varprojlim M_i$ compact

$\varinjlim$ is exact, but $\varprojlim$ is only left exact

$\left(\begin{array}{l} \text{if} \\ 0 \to M_i' \to M_i \to M_i'' \to 0 \ \forall i \in I, \\ \\ 0 \to \varinjlim M_i' \to \varinjlim M_i \to \varinjlim M_i'' \to 0 \end{array} \right)$

if $n | m$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "m"] \arrow[u, "\frac{m}{n}"] & \mathbb{Z} \arrow[r] \arrow[u, "1"'] & \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[u, "proj"'] & 0 \end{tikzcd}$$

$0 = \lim \mathbb{Z}$

$\mathbb{Z} \to \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$

Top gps.

G ab gp under $+$

G top space with

$G \times G \longrightarrow G$ continuous

T_1: points are closed

T_2: $\forall P, Q$ $\exists$ open set $U \subset P$ $V \subset Q$ with $U \cap V \overset{?}{=} \phi$

T_0: $\forall P, Q$, eith $\exists$ open subset $U \subset P$, $Q \notin U$, or $V \subset Q$, $P \notin V$

Ex. $\mathbb{Z}$ finite sets are closed

$\mathbb{Z} \times \mathbb{Z} \xrightarrow{x-y} \mathbb{Z}$ $\quad \Delta = (x-y)^{-1}(0)$

$\Delta \subseteq \mathbb{Z} \times \mathbb{Z}$ closed? diagonal

metric $\Rightarrow$ Hausdorff (T_2)

Spec A is T_0, with closed sets $V(I) = \{P \supseteq I\}$

Prop. G top. gp., T_1 $\Rightarrow$ G Hausdorff

pf. $\Delta = (x-y)^{-1}(0)$

$||$

$G \times G \xrightarrow{x-y} G$

$\{0\}$ closed in G $\Rightarrow$ Δ closed in $G \times G$

Δ closed in $X \times X \Leftrightarrow X$ Hausdorff

Atiyah Macdonald don't assume their top. gps are T_1

G top gp. let $H = \bigcap_{\substack{U \ni e \\ U \text{ open in } G}} U$. Then H subgroup of G.

Pf. $x \in$ every open neighborhood of G

$y \in$ ''

$G \times G \xrightarrow{\phi} G$ cont.

$(x,y) \mapsto x+y$

$\phi^{-1}(0) \ni (0,0)$

W open in G, $W \ni 0$

$\phi^{-1}(W)$ open in $G \times G$ $\exists U, V : U \times V < \phi^{-1}(W)$ $U \ni 0$, $V \ni 0$

$x \in U$, $y \in V$, $(x,y) \in \phi^{-1}(W)$

$\phi(x,y) \in W$ x+y

Prop. a) $H = \overline{0}$

b) G/H is Hausdorff with quotient topology

U open in G/H $\Leftrightarrow$ $\pi^{-1}(U)$ open in G

$\pi: G \to G/H$

C closed in G/H $\Leftrightarrow$ $\pi^{-1}(C)$ closed in G

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$\{0\}$ closed in G/H because operation is homeomorphism,

all $\{a\}$ closed in G/H; thus G/H Hausdorff

G Hausdorff $\Leftrightarrow$ $H=0$

Cauchy sequences

G top ab gp.

$\{x_n\}$ is Cauchy if $\forall U$ neighborhood of 0, $\exists N: n,m > N \Rightarrow$

$x_n - x_m \in U$

Equiv classes $\{x_n\} \sim \{y_n\}$ if $\forall U \ni 0$

$\exists N: n > N \Rightarrow x_n - y_n \in U$

prove this is equiv relation

define op: $\{x_n\} \sim \{x'_n\}$, $\{y_n\} \sim \{y'_n\} \Rightarrow \{x_n + y_n\} \sim \{x'_n + y'_n\}$

$\hat{G}$ = equiv classes of Cauchy sequences

$\hat{G}$ top gp   System of neighborhoods of 0 $\{x_n\}: x_n \in U$ for $n \gg 0$

U neighborhood in G

$G \xrightarrow{\phi} \hat{G}$

Kernel of ϕ is $H = \bigcap$ all neighborhoods of 0

Let G be A^+ for ring A

Let U_n subgroups of G be I^n, powers of ideal I in A

equiv classes of Cauchy seq.

$\hat{G} = \hat{A} = \varprojlim A/I^n$ with product top

$x \in \varprojlim A/I^n \subseteq \prod A/I^n$

$x = \{\bar{x}_n\}$ $x_n \in A$, $\bar{x}_n \in A/I^n$ $n > m$, $\pi_{n,m}: \pi_{n,m}(\bar{x}_n) = \bar{x}_m$

$x_n - x_m \in I^m$

Claim. 1) $\{x_n\}$ is a Cauchy sequence

2) If $x = \{\overline{x_n}\} = \{\overline{y_n}\}$, then $\{x_n\} \sim \{y_n\}$, $x_n - y_n \in I^n$

$\{z_n\}$ Cauchy-sequence of elements of A

take $\overline{z_n} \in A/I^n$

Can take subsequence $n \geq m$ $z_n - z_m \in I^m$

eventually $z_n - z_m \in I$

Let $w_1 = z_N$ $\qquad$ $n, m \geq N_1$ $\quad z_n - z_m \in I$

$w_2 = z_{N_2}$ $\qquad$ $n, m \geq N_2$ $\quad z_n - z_m \in I^2$

$\vdots$

Lecture 19 (2009-04-09)

4/9/09

Prop. An open subgroup H of a top. gp. is closed (also works for non-comm)

$G = \bigsqcup gH$ left cosets

$x \mapsto gx$ is a homeomorphism, and so preserves open sets

So H open $\Rightarrow gH$ open; thus $\bigcup_{gH \neq H} gH$ is open (union of open sets)

but then $H = G - \bigcup_{gH \neq H} gH$ is closed.

Let k_s = sep closure of k field

$\underbrace{k \subseteq k'}_{\substack{\text{normal} \\ \text{finite}}} \subseteq k_s$ Let $G_{k'} = Gal(k'/k)$

$k_1 \geq k_2$ if $k_1 \supseteq k_2$

$G_{k_1} \twoheadrightarrow G_{k_2}$ directed set

$\varprojlim_{k'} Gal(k'/k) \overset{def}{=} Gal(k_s/k)$ compact, Hausdorff top. gp.

profinite, by definition

k finite field

$$\begin{tikzcd} k_n \arrow[d, dash, "n"'] \\ k_1 \end{tikzcd}$$

$k_n \subseteq k_m \iff n \mid m$

$G(k_n/k) = \mathbb{Z}/n\mathbb{Z}$

$Gal(\bar{k}/k) = \hat{\mathbb{Z}} = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})$

Now we'll use Atiyah/Macdonald's def of inverse system: inverse system with $I = \mathbb{N}$

$G_{n+1} \to G_n \to G_{n-1} \cdots$

Everything is coherent automatically

If $\{A_n\}, \{B_n\}, \{C_n\}$ are inverse systems

Maps are
$$\begin{tikzcd} A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\theta_{n+1,A}"'] & B_{n+1} \arrow[d, "\theta_{n+1,B}"] \\ A_n \arrow[r, "f_n"] & B_n \end{tikzcd}$$

$0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ exact iff

$0 \to A_n \to B_n \to C_n \to 0$ exact $\forall n$

Prop. If $0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ exact,

$A = \varprojlim A_n$, $B = \varprojlim B_n$, $C = \varprojlim C_n$, then

$0 \to A \to B \to C$ exact, and if $\{A_n\}$ is a surjective system (θ_n's are surjective)

Then $0 \to A \to B \to C \to 0$ exact

$\varprojlim A_n \subseteq \prod A_n$ consists of $\{a_n\}$ with $\theta_{n+1}(a_{n+1}) = a_n$

Pf. Let $\tilde{A} = \prod A_n$, $\hat{B} = \prod B_n$, $\hat{C} = \prod C_n$

Consider

$$\begin{tikzcd}
0 \arrow[r] & \tilde{A} \arrow[r] \arrow[d, "d_A"] & \tilde{B} \arrow[r] \arrow[d, "d_B"] & \hat{C} \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & \tilde{A} \arrow[r] & \tilde{B} \arrow[r] & \hat{C} \arrow[r] & 0
\end{tikzcd}$$

$d_A : \tilde{A} \to \tilde{A}$, $d_A\{a_n\} = \{a_n - \theta_{n+1}(a_{n+1})\}$

d_B, d_C similarly

By snake lemma, $0 \to \mathrm{Ker}(d_A) \to \mathrm{Ker}(d_B) \to \mathrm{Ker}(d_C) \to \mathrm{Coker}(d_A) \to \mathrm{Coker}(d_B) \to \mathrm{Coker}(d_C) \to 0$

but $\mathrm{ker}(d_A) = A$ because any $\{a_n\}$ for which $a_n - \theta_{n+1}(a_{n+1}) = 0$ $\forall n$ is in the inverse limit.

Thus we have $0 \to A \to B \to C$ exact

If $\{A_n\}$ surjective, $\mathrm{Coker}(d_A) = 0$, because

d_A is surjective: $\{x_n\} : \{x_n - \theta_{n+1}(x_{n+1})\} = \{a_n\}$

Take any x_1 define x_{n+1} by $\theta_{n+1}(x_{n+1}) = x_n - a_n$

In some sense, $\mathrm{Coker}(d_A)$ is a derived functor of $\varprojlim$, because we have

$0 \to A \to B \to C \to \mathrm{Coker}(d_A) \to \mathrm{Coker}(d_B) \to \mathrm{Coker}(d_C) \to 0$

$\mathrm{Coker}(d_A) = \varprojlim{}^1 \{A_n\}$, vanishes if $\{A_n\}$ surjective

If G top gp, G_n decreasing sequence of subgps

$G = G_0 \supseteq G_1 \supseteq \cdots$

Define $\hat{G} = \varprojlim_n G/G_n$

$G \to G/G_n$

$G/G_{n+1} \xrightarrow{\theta_{n+1}} G/G_n \xrightarrow{\theta_n} G/G_{n-1}$

4/9/09

If $0 \to G' \xrightarrow{\psi} G \xrightarrow{\phi} G'' \to 0$ exact

G_n subgps of G

Let $G'_n = G' \cap G_n$ (inverse image really)

$G''_n = \phi(G_n)$

$$0 \to G'/G'_n \to G/G_n \to G''/G''_n \to 0$$

Corollary $0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$ exact

A a ring, I ideal

$A \supseteq I \supseteq I^2 \supseteq \cdots$

Map out of direct system

Map into inverse system

M A-module

$M \supseteq IM \supseteq I^2M \supseteq \cdots$

$\hat{M} = \varprojlim_n M/I^nM$ module over $\hat{A} = \varprojlim_n A/I^nA$

Thm. $0 \to M_1 \to M_2 \to M_3 \to 0$ exact A-modules ⟸ we'll prove this later

$\Rightarrow 0 \to \hat{M}_1 \to \hat{M}_2 \to \hat{M}_3 \to 0$ exact $\hat{A}$-modules

Why isn't this the same as our thm for gps?

Because we're computing wrt I^nM_1, i.e. $M_{1,n} = I^nM_1$, whereas $G'_n = I^nM_2 \cap M_1$; not necessarily the same

Let $G' = G_n$

$$0 \to G_n \to G \to \overset{G''}{\overset{\parallel}{G/G_n}} \to 0$$

G'' discrete top

$$0 \to \hat{G}_n \to \hat{G} \to \hat{G}'' \to 0$$

$\hat{G}/\hat{G}_n \simeq G/G_n$

So $\widehat{(\hat{G})} \simeq \hat{G}$

Def. G complete if $G \to \hat{G}$ is an isomorphism

Cor. $\hat{G}$ is complete

Ex. $A = k[x]$

$I = (x)$

Then $\hat{A} = k[[x]]$

$$\begin{tikzcd}
k[[x]] \arrow[r] \arrow[dr] & A/I^n \\
 & A/I^{n+1} \arrow[u]
\end{tikzcd}$$

$k[[x]] \simeq \varprojlim k[[x]]/x^n$ ✓

A filtration of M is a decreasing sequence of A-submodules of M

$M = M_0 \supseteq M_1 \supseteq M_2$ ---

Ex. $M = A$, $M_n = I^n$ } stable

$M = M$, $M = I^n M$ } I-filtrations

I-filtration is if $IM_n \subseteq M_{n+1}$

Stable I-filtration if $IN_n = M_{n+1}$ for all $n \geq$ some N.

Lemma. Suppose $\{N_n\}$, $\{M_n'\}$ are stable I-filtrations. Then they have <u>bounded difference</u>, i.e. $\exists\, n_0 : M_{n+n_0} \subseteq M_n'$

$M_{n+n_0}' \subseteq M_n$

Pf. If $\{a_n\}$, $\{b_n\}$ have bd. difference, $\{b_n\}$, $\{c_n\}$ have bd. difference, then $\{a_n\}$, $\{c_n\}$ have bd. difference, i.e. equiv relation

So suffices to show

$\{M_n\}$ and $\{I^n M\}$ have bd difference

M_n is an I-filtration

$I^n M = I^n M_0 \subseteq M_n$

$IM_n = M_{n+1}$ for $n \geq N$

$I^n M_N = M_{N+n}$

$M_{N+n} = I^n M_N \subseteq I^n M$

4/9/09

Graded Modules + rings

A graded ring if $A = \bigoplus_{n \geq 0} A_n$ A_n abelian $\overset{\text{sub}}{\text{groups}}$ of A

$A_m A_n \subseteq A_{m+n}$

So every $y \in A$ is $y = y_0 + \cdots + y_n$

M graded A-module $M = \bigoplus_{n \geq 0} M_n$ M_n subgps of M

$A_m M_n \subseteq M_{m+n}$ A_n, M_n are A_0-modules

An element $y_n \in A_n$ is homognous

eg. $A = k[x_1, \ldots, x_r]$ A_n = homogeneous polynomials of total degree n

$\deg(x_1^{a_1} \cdots x_r^{a_r}) = a_1 + \cdots a_r$

A $\mathbb{Z}$-graded ring is $A = \bigoplus_{n \in \mathbb{Z}} A_n$, $A_n A_m \subseteq A_{n+m}$

A graded ring, f homogenous element

A_f is a $\mathbb{Z}$-graded ring we want $\deg(a/f^n) = \deg(a) - n\deg(f)$

$\mathbb{Z}[[x]]_x$ = Laurent polynomials

so localizing a graded ring isn't necessarily graded, but is $\mathbb{Z}$-graded

Def. A homomorphism of graded A-modules

$M \xrightarrow{f} N$ f is map of A-modules

$N_n \xrightarrow{f_n} N_n$ $f(M_n) \subseteq N_n$

Thm. Let A be a graded ring, TFAE

1) A is noeth (as ring)
2) A_0 is noeth, and A is f.g. A_0-algebra

Pf. 2) $\Rightarrow$ 1) by properties of noeth

Let $A_+ = \bigoplus_{n > 0} A_n$ ideal in A

$A/A_+ \simeq A_0$ $n > 0$ $y \in A_n$, $y = \sum a_i x_i$ $a_i \in A_{n-k_i}$ $n-k_i < n$

Let $A_+ = (x_1, \ldots, x_r)$

if A_+ generated by x_i's, it is generated by their homogenous components, so WLOG we let the x_i be homogeneous

Let x_i be of deg. k_i

Let $A' = A_0[x_1, \ldots, x_r]$. Claim: $A_n \subseteq A' \forall \Rightarrow A = A'$

Lecture 20 (2009-04-14)

4/14/09

A noeth ring, I ideal, M f.g. A mod

$\hat{A}_I = \varprojlim_n A/I^n$ we will eventually prove

- $\hat{A}_I \otimes_A M \xrightarrow{\lambda_M} \hat{M}_I$ λ_M is isomorphism
- $0 \to M' \to M \to M'' \to 0$ exact, each f.g.

$\Rightarrow 0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$ exact (i.e., $\hat{A}_I$ flat $/A$)

- A noeth $\Rightarrow \hat{A}_I$ noeth

A any ring, I ideal

$A^* = \bigoplus_{n \geq 0} I^n$, def $I^0 = A$

graded ring, with $I^n \otimes I^m \underset{\text{multiplication}}{\to} I^{n+m}$

similarly, if M A-module, M_n I-filtration ($IM_n \subseteq M_{n+1}$)
$M_0 = M$

$M^* = \bigoplus_{n \geq 0} M_n$ M^* is a graded A^* module

$I^n \otimes M_k \to M_{n+k}$

A noeth $\Rightarrow$ I is f.g., $= (x_1, \ldots, x_r) \Rightarrow$
A^* is noeth (generated over A by $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_r$) because is quotient of
$A[X_1, X_2, \ldots, X_r]$.

Lemma. Let A noeth, M f.g. A-module.
M_n an I-filtration of M. Then TFAE:
1) M^* f.g. A^*-module
2) $\{M_n\}$ stable I-filtration

Pf. Let $Q_n = \bigoplus_{r=0}^{n} M_r$, subgroup of M^* (but not subring since not closed under mult)

Let $M_n^* = Q_n \oplus IM_n \oplus I^2M_n \oplus \cdots$, this is an A^*-module

$M_n^* \subseteq M^*$ by construction

M_n^* is a f.g. A^*-module (generated by Q_n) because $M_n \subseteq Q_n$ and so everything is either in Q_n or a product of something in $I \subset A^*$ and something in Q_n

$\bigcup_n M_n^* = M^*$

A^* noeth $\Rightarrow$ (M^* f.g. A^*-module $\Leftrightarrow$ chain stops $\Leftrightarrow$ $M_{n_0+r} = I^r M_{n_0}$ some n_0, $\forall r \geq 0$)

$\Updownarrow$

def. of. stable I-filtration

Prop (Artin-Rees)

A noeth, I ideal, M f.g. A-module,

$\{M_n\}$ stable I-filtration of M

If $M' \subseteq M$ is a sub A-module, then $\{M_n \cap M'\}$ is stable

Pf. $I(M_n \cap M') \subseteq IM_n \cap IM' \subseteq M_{n+1} \cap M'$, so I-filtration

so we get graded A^*-module $(M')^* \subseteq M^*$, and because M^* is noeth, $(M')^*$ is f.g., and by lemma, $\{M_n \cap M'\}$ stable I-filtration

---

Now let $M_n = I^n M$, M^* f.g. A^* module

$\exists k : I^n M \cap M' = I^{n-k}(I^k M \cap M')$ $\forall n \geq k$ $(k = n_0)$

Thm. A noeth M f.g. A-module, I ideal of A

M' a sub A-module of M, then $I^n M'$ and $I^n M \cap M'$ have bounded difference

Pf. ✓ ([illegible])

This implies $\hat{M}'_{(I^n M \cap M')} \simeq \hat{M}'_{I^n M'}$

because bdd difference $\Rightarrow$ induces same topology on completion

So completion is exact functor on modules / noeth rings

$0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow$ $0 \to \underset{\hat{M}'_{I^n M'}}{\underset{\wr\!\mid}{\hat{M}'_{(I^n M \cap M')}}} \to \hat{M}_{I^n M} \to \hat{M}''_{I^n M''} \to 0$ exact

by completion lemma

4/14/09

A ring, I ideal, M A-module $\quad \hat{M} = \varprojlim M/I^n M$

$\hat{A} = \varprojlim A/I^n$

$\exists$ map $\hat{A} \otimes_A M \xrightarrow{\lambda_m} \hat{M}$

Thm. If M is f.g. the λ_m is surjective

M is finitely presented, (f.p.), λ_m is bijective

(f.g means can write $A^r \xrightarrow{f} M \to 0$,

f.p means ker(f) also f.g., i.e. set of relations is f.g. so write $A^s \to A^r \to M \to 0$)

A Noeth $\Rightarrow$ (M f.g. $\Rightarrow M$ f.p) because ker(f) is submodule of A^r and hence is f.g.

Certainly, if $M' = A$, $\hat{A} \otimes_A A' \xrightarrow{id} \hat{A}$ since $\hat{A} \otimes_A A = \hat{A}$, so λ_A is identity, which implies λ_{A^r} is identity, but $A^s \to A^r \to M \to 0$ exact, so

$$\begin{tikzcd}
A^r \otimes_A \hat{A} \arrow[r] \arrow[d, "\lambda_M"] & M \otimes_A \hat{A} \arrow[r] \arrow[d, "\lambda_M"] & 0 \\
\hat{A}_r \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

$\widehat{\bigoplus_I M_i} \neq \bigoplus_I \hat{M}_i$

if I is infinite

$$\begin{tikzcd}
A^s \otimes \hat{A} \arrow[r] \arrow[d, "\lambda", "\wr"'] & A^r \otimes \hat{A} \arrow[r] \arrow[d, "\lambda", "\wr"'] & M \otimes \hat{A} \arrow[r] \arrow[d, "\lambda_m", "\wr"'] & 0 \\
\hat{A}^s \arrow[r] & \hat{A}^r \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

λ_m is id by 5 lemma

Thus, if A noeth, $\_\_ \otimes_A \hat{A}$ exact on f.g. modules

If M f.g. A module, with $0 \to R \to F \to M \to 0$, so we get

$\underline{0} \to \operatorname{Tor}_1^A(M, \hat{A}) \to R \otimes_A \hat{A} \to F \otimes_A \hat{A} \to M \otimes_A \hat{A} \to 0$, but by exactness we get this, but then $\operatorname{Tor}_1^A(M, \hat{A}) = 0$; Tor commutes with limits, so

$M = \lim M_\alpha$, M_α f.g., $\operatorname{Tor}_1^A(M, \hat{A}) = \varinjlim \operatorname{Tor}_1^A(M_\alpha, A_\alpha) = 0$

$\Rightarrow \hat{A}$ flat $/A$

Prop. Assume A noeth, I ideal

1) $\hat{I} \simeq I\hat{A} \simeq \hat{A} \otimes_A I$
2) $\widehat{(I^n)} \simeq (\hat{I})^n$
3) $I^n/I^{n+1} \simeq (\hat{I})^n/(\hat{I})^{n+1}$
4) $\hat{I} \subseteq$ Jacobson radical $(\hat{A})$

Pf. $\widehat{(I^n)} \simeq I^n\hat{A} \simeq (I\hat{A})^n = (\hat{I})^n$

Cheating somewhat; $\widehat{(I^n)}$ is really completion w.r.t. I^n, not I, but because they make same top, its okay

$G/G_n \simeq \hat{G}/\hat{G}_n$, so $A/I^n \simeq \hat{A}/\widehat{(I^n)} \simeq \hat{A}/(\hat{I})^n$

so $\hat{A}$ is complete for $\hat{I}$-topology

Let $x \in I$; then $(1-x)^{-1} = 1 + x + x^2 + \cdots$ formally, but this actually converges in $\hat{A}_I$, so $x \in I \Rightarrow 1-x$ invertible, so $x \in$ Jacobson radical.

Corollary. If I is max ideal of A, then $\hat{A}$ is local ring with $\hat{I}$ max ideal

$\underset{\text{field}}{\overset{\shortparallel}{A/I}} \simeq \hat{A}/\hat{I}$, so $\hat{I}$ max, but $\hat{I} \subseteq$ Jacobson radical, and since $\hat{I}$ is maximal, must have $\hat{I} =$ Jacobson radical, so $\hat{A}$ local

Example. $\mathbb{Z} = A$
$I = (p)$

$$\hat{\mathbb{Z}}_{(p)} = \mathbb{Z}_p = \widehat{(\mathbb{Z}_{(p)})}_{p\mathbb{Z}_{(p)}}$$

localize then complete = complete

4/14/09

Thm. A noeth, I ideal, M f.g. A-mod

Let $E = \bigcap_n I^n M = \mathrm{Ker}(M \to \hat{M})$

$$\overset{\shortparallel}{\varprojlim} M/I^nM \subseteq \prod M/I^nM$$

kernel of $M \to \prod M/I^nM$ must be in kernel of each, i.e. in each I^n

E consists of those $x \in M$ annihilated by some element of $1+I$

Induced topology on E is trivial, $I^nM \cap E = E \Rightarrow IE = E$

E f.g. $\Rightarrow (1-\alpha)E = 0$ for some $\alpha \in I$

If $(1-\alpha)x = 0 \quad x \in M \quad \alpha \in I$

$x = \alpha x = \alpha^2 x = \alpha^3 x \cdots$

$x \in I^nM \ \forall n$, $x \in \bigcap I^nM = E$

Cor. A noeth domain $\Rightarrow \bigcap_n I^nA = 0$ since nothing annihilates anything

Cor. M f.g., m max ideal of noeth local ring A

$\bigcap_n m^n M = 0$, also $\bigcap_n m^n = 0$

Next time: A ring, I ideal

associated graded ring $G_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1} =$

$A/I \oplus I/I^2 \oplus \cdots$

$A = k[[x_1, \cdots x_n]]$

$I = (x_1, \ldots, x_n)$

$G_I(A) = k[\underbrace{\bar{x}_1, \ldots, \bar{x}_n}_{\in I/I^2}]$

Lecture 21 (2009-04-16)

HW Review 4/16/09

M torsion $\Leftrightarrow \mathrm{Tor}_1(M, \mathbb{Q}/\mathbb{Z}) = 0$

$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$\mathrm{Tor}_1(M, \mathbb{Q}) \to \mathrm{Tor}_1(M, \mathbb{Q}/\mathbb{Z}) \to M \to \mathbb{Q} \otimes M \to \mathbb{Q}/\mathbb{Z} \otimes M \to 0$

$\mathbb{Q}$ is a localization of $\mathbb{Z}$, hence flat

$$\begin{array}{c} \mathrm{Ker}(M \to \mathbb{Q} \otimes M) = \text{Torsion submodule of } M \\ \shortparallel \\ \mathrm{Tor}_1(M, \mathbb{Q}/\mathbb{Z}) \end{array}$$

$m/1 = 0 \Leftrightarrow \exists t \neq 0 : tm = 0$

A a valuation ring, value gp. G, determine the ideals of A e.g. $G \subset \mathbb{R}$

ideals = subsets of G where $x \in S$, $y \geq x \Rightarrow y \in S$

$G_+ = \{x \in G : x \geq 0\}$

Let $I = \{x : v(x) \in S\}$. $x, y \in I$

$v(x+y) \geq \min(v(x), v(y)) \in S$

$v(ax) = v(a) + v(x) \geq 0 + v(x) = v(x) \in S$

I, let $S = \{v(I)\}$

$x \in S \quad y \geq x$

$v(a) = x$, $v(b) = y$

$v(ba^{-1}) \geq 0$, so $ba^{-1} \in A$

$y \in I \quad v(y) \in S$

So if $G = \mathbb{R}$

$\{x : x \geq a \text{ for some } a \geq 0\}$

$\{x : x > a \text{ for some } a \geq 0\}$

$v(a) \geq 0$, so $v(I)$ bounded below, but inf not necessarily in set.

If $G = \mathbb{R}$, what are prime ideals?

0, and $\{x : v(x) > 0\} = m$,

but if I prime, $y \in I \neq 0$

$x \in m$, $\exists n : v(x^n) = n v(x) > v(y) \quad x^n \in I \Rightarrow x \in I$

so these are only

If $G = \mathbb{R} \oplus \mathbb{R}$,
0, $\{x : v(x) > 0\}$
and $\{x : v(x) \geq (a,b)\}$
$a > 0$, b arbitrary?

Valuation ring + noeth $\Rightarrow$ DVR

Every f.g. ideal is principal

$(x_1, x_2, \dots, x_n)$ gen by x_i : $v(x_i)$ minimal

PID . local

M f.g. A-module

A noeth ring

$\exists x \in M : x \neq 0$, $\text{Ann}(x) = P$ prime

Pf. Look at $\{\text{Ann}(y), y \in M\}$.

$\exists$ max element. $\text{Ann}(x) = P$. This is prime because

$ab \in P$ $b \notin P$ $bx \neq 0$ $a(bx) = 0$

$\text{Ann}(bx) \supseteq P$

$a \in \text{Ann}(bx)$. But P maximal, so must have $a \in P$

So get $0 \to A/P \to M \to M_1 \to 0$

$1 \mapsto x$
$a \mapsto ax$

Do same thing with M_1

$0 \to A/P_1 \to M_1 \to M_2 \to 0$

$\vdots$

Get $N_1 = \ker(M \to M_1)$
$N_2 = \ker(M \to M_2)$

$\vdots$

$N_1 \subseteq N_2 \subseteq$, Stops.

So write M as sum of (A/P)'s

Get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & \bigoplus_P M'_P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \bigoplus_P M_P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'' \arrow[r] \arrow[d] & \bigoplus_P M''_P \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

$(A/Q)_Q = A/Q$

$(A/Q)_P = 0$ for $P \neq Q$

only true for max Q, but in Dedekind domain prime $\Rightarrow$ max

A Dedekind domain

$I = A/(y)$ principal

so $\bar{I} = (x)$

$I = (x, y)$.

<u>Associated graded rings</u>

A ring I ideal

$G_I(A) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ $\quad I^0 = A$

$\{M_n\}$ I-filtration of M, an A-module

$G(M) = \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$ is a $G_I(A)$-module

Prop. If A noeth, $G_I(A)$ noeth.

Pf. A noeth $\Rightarrow I = (x_1, \dots, x_n)$, and $G_I(A)$ is a quotient of $A[X_1, \dots, X_n]$ which is noeth by Hilbert basis thm (images of generators of I generate I^n/I^{n+1} $\forall n$)

Prop. $G_I(A) \simeq G_{\hat{I}}(\hat{A})$

Pf. $I^n/I^{n+1} \simeq \hat{I}^n/\hat{I}^{n+1}$

Prop. M f.g. A-module

M_n stable I-filtration

$\Rightarrow G(M)$ f.g. $G_I(A)$-module

Pf $G(M)$ generated by $\bigoplus_{n \leq n_0} G_n(M)$, a f.g. A/I module

$n \geqslant n_0 \quad I^{n-n_0} M_{n_0} = M_n$

$G_n(M) = M_n / M_{n+1}$

Lemma. $\phi : A \to B$ homomorphism of filtered (ab) grps.

$\phi(A_n) \subseteq B_n$

$G(\phi) : G(A) \to G(B)$

$\bigoplus A_n/A_{n+1} \to \bigoplus B_n/B_{n+1}$

$\hat{\phi} : \varprojlim A/A_n \to \varprojlim B/B_n$

Thm. $G(\phi)$ injective $\Rightarrow \hat{\phi}$ injective

$G(\phi)$ surjective $\Rightarrow \phi$ surjective

Pf.

$$\begin{tikzcd}
0 \arrow[r] & A_n/A_{n+1} \arrow[r] \arrow[d] & A/A_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & A/A_n \arrow[r] \arrow[d, "\alpha_n"] & 0 \\
0 \arrow[r] & B_n/B_{n+1} \arrow[r] & B/B_{n+1} \arrow[r] & B/B_n \arrow[r] & 0
\end{tikzcd}$$

this 0 and this 0 $\Rightarrow$ this 0

$0 \to \operatorname{Ker} G_n(\phi) \to \operatorname{Ker} \alpha_{n+1} \to \operatorname{Ker} \alpha_n \to \operatorname{Coker} G_n(\phi) \to \operatorname{Coker} \alpha_{n+1} \to \operatorname{Coker} \alpha_n \to 0$

induction $\Rightarrow$ ($\operatorname{Ker} \alpha_n = 0 \ \forall n \Rightarrow \hat{\phi}$ injective, since $\varprojlim$ left exact)

and

induction $\Rightarrow \operatorname{Coker} \alpha_n = 0$ because

$\operatorname{Ker} \alpha_n \to A/A_n \xrightarrow{\alpha_n} B/B_n \to 0$

$\varprojlim$ not in general surjective, but because maps $\operatorname{Ker} \alpha_{n+1} \to \operatorname{Ker} \alpha_n$ are surjective

get from snake lemma that $\varprojlim$ right exc.

4/16/09

Prop A ring, I ideal M A-module

M_n I-filtration of M, assume $A \simeq \varprojlim A/I^n = \hat{A}$

M_n Hausdorff (so that $\bigcap_n M_n = 0$)

<u>Thm</u> $G(M)$ is an f.g. $G_I(A)$-module $\Rightarrow$ M f.g. A-module

Pf. pick finite set $\overline{x_i}$ of generators of $G(M)$. Assume $\overline{x_i}$ homogeneous

$x_i \in M_{n(i)} \mapsto \underset{\overline{x_i}}{\underset{\cup\!\!\!\!\!\!|}{M_{n(i)}}} / M_{n(i+1)}$ x_i for $i = 1, \ldots, r$

Let F^i = module A with stable I-filtration

given by $F^i_k = I^{k-n(i)}$ for $k \geq n(i)$

$F^i_k = A$ for $k \leq n(i)$

ATIYAH MACDONALD GETS THIS WRONG

$$\begin{tikzcd} 1 \arrow[d, mapsto] & A \arrow[d] & \geq A \arrow[d] & \geq A \arrow[d] & \cdots & \supseteq A \arrow[d] & \supseteq I \arrow[dr] & \supseteq I^2 \supseteq \cdots \\ x_i & M = M_0 & \geq M_1 & \geq M_2 & \cdots & \supseteq M_{n(i)} & & \geq M_{n(i+1)} \end{tikzcd}$$

$\begin{pmatrix} \downarrow \\ 0 \end{pmatrix}$ $\left(M_0/M_1 \right)$ $\left(M_1/M_2 \right)$

because $x_i \in M_{n(i)}$

$1 \mapsto x_i$

$F^i \to M$

$F \twoheadrightarrow M$

Let $F = \bigoplus_{i=1}^{r} F^i$

$G(F) \xrightarrow{G(\phi)} G(M)$

$\overline{1} \mapsto \overline{x_i}$

A <u>complete</u> by hypothesis

so F complete, i.e. $F \simeq \hat{F}$

By construction $G(\phi)$ surjective, so $\hat{\phi}$ surjective $\hat{\phi} : \hat{F} \to \hat{M}$

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, squiggly, "\alpha"'] & M \arrow[d, "\beta"] \\ \hat{F} \arrow[r, "\hat{\phi}"] & \hat{M} \end{tikzcd}$$

$\operatorname{Ker}(\beta) = \bigcap M_n = 0$

so β injective, surjective

so β iso, so ϕ surjective

$\Rightarrow$ M is a f.g A-module. s/he s/b of free

Same assumptions

Cor. $G(M)$ noeth, $G(A)$-module $\Rightarrow$ M noeth A module

Pf. Let $M' \subseteq M$, let $M'_n = M_n \cap M'$

M'_n is I-filtration of M' $\quad M' \hookrightarrow M \quad M'_n/M'_{n+1} \hookrightarrow M_n/M_{n+1}$

$M'_n \hookrightarrow M_n \quad G(M') \hookrightarrow G(M)$

M' Hausdorff, so $\bigcap M'_n = 0$ $\Rightarrow$ $G(M')$ noeth so f.g.

Prop $\Rightarrow$ M' is f.g.

Prop. A noeth, I ideal, $\Rightarrow$ $\hat{A}_I$ is noeth

Pf. $G_I(A)$ noeth, so $\hat{A}_I$ complete, so $\hat{A}_I \overset{\chi}{\xrightarrow{\sim}} \hat{\hat{A}}_{\hat{I}}$ is isomorphism

so $Ker(\chi) = \bigcap_n \hat{I}^n = 0$, so A, with I-filtration, is Hausdorff $\Rightarrow$ $\bar{A}$ noeth

Lecture 22 (2009-04-21)

4/21/09

Dimension Theory

why are x_i of deg 1 homogeneous

graded noeth ring

$A = \bigoplus A_n$, $A_n A_m \subset A_{n+m}$

w/ A_0 noeth, A is a f.g. A_0-algebra

Choose generators $x_1, \ldots, x_s$ for A as an A_0-algebra, WLOG take them to be homogeneous, say $x_i \in A_{k_i}$ (we write $\deg(x_i) = k_i$ in the grading)

Let $M = \bigoplus M_n$ be an f.g. graded A-module, $A_i M_j \subset M_{i+j}$

Choose generators for M, $m_1, m_2, \ldots m_t$ homogeneous, $m_j \in M_{r_j}$ $\deg(m_j) = r_j$

Claim: M_n is an f.g. A_0-module

Pf. $m \in M_n$, so m is homogeneous of deg n.

$$m = \sum_{j=1}^{t} \left(\begin{smallmatrix}\text{element}\\ \text{of } A\end{smallmatrix}\right) \cdot m_j = \sum_{j=1}^{t} \left(\begin{smallmatrix}\text{poly in } x_1 \ldots x_s\\ \text{with } A_0\text{-coefs}\end{smallmatrix}\right) \cdot m_j$$

$$\sum (A_0^{elt}) x_1^{e_1} \cdots x_s^{e_s}$$

each needs to be of deg n. So deg of poly needs to be $n - r_j$

deg of this is $e_1 k_1 + \cdots + e_s k_s$

So we need $e_1 k_1 + \cdots e_s k_s = n - r_j$

So M_n as an A_0-module is generated by $x_1^{e_1} \cdots x_s^{e_s} m_j$ with

$1 \leq j \leq t$ and $e_i \geq 0$, $\sum_{i=1}^{s} e_i k_i = n - r_j$

Def. Let $\lambda : \{\text{f.g. } A_0\text{-modules}\} \to \mathbb{Z}$ be additive, i.e. if

$0 \to M \to N \to P \to 0$ (exact) then $\lambda(M) - \lambda(N) + \lambda(P) = 0$. (Equivalently,

$0 \to M_1 \to M_2 \to \cdots \to M_k \to 0$ has $\sum_{i=1}^{k} \lambda(M_i)(-1)^i = 0$ — split long seq. up into short. Also, this is like an Euler characteristic)

$\lambda(0 \text{ module}) = 0$ $\forall \lambda$ since $0 \to 0 \to 0 \to 0 \to 0$

Example: A_0 = field

$\lambda(M) = \dim(M)$ as A_0-vector space

ONLY POSSIBLE NON-TRIVIAL λ

A_0 = PID

M = finite $\times A_0^r$

$\lambda(M) = r$

M_n is a f.g. A_0-module

$\lambda(M_0), \lambda(M_1) \ldots$

What do we do with sequences?

Poincaré series of M is

$$P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n) T^n$$

If A_0 field, $M_n = A_0^n$,

$$\sum_{n=0}^{\infty} n T^n = \frac{T}{(1-T)^2}$$

Thm (Hilbert, Serre)

$P(M,t) \in \mathbb{Q}(t)$

$$P(M,t) = \frac{\text{poly in } \mathbb{Z}[t]}{(1-t^{k_1}) \cdots (1-t^{k_s})}$$

Special case: $\deg x_i = 1 \ \forall i$

$$P(M,t) = \frac{\text{polynomial}}{(1-t)^s}$$ where s = # of generators

Pf. Induction on S = # of generators of A as A_0-algebra

$\boxed{}$ $s=0$, $A = A_0$, so $A_n = 0 \ \forall n \geq 1$

M is f.g. A-module, but $A = A_0$, so M is f.g. A_0-module.

So $M_n = 0$ for sufficiently large n, say $n > n_0$

So $P(M,t) = \sum_{n=0}^{n_0} \lambda(M_n) t^n$ since $\lambda(0) = 0$

Assume true for $s-1$, assume $\deg x_s = 1$

look at $M \to M$

$m \mapsto x_s m$

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would be L_{n+k_s} if $\deg(x_s)=k_s$

$$0 \to K_n \to M_n \xrightarrow{m \mapsto x_s m} M_{n+1} \to L_{n+1} \to 0$$

K_n kernel, L_{n+1} cokernel

Let $K = \oplus K_n$
$L = \oplus L_n$
graded

$$t^{n+1}\lambda(K_n) - t^{n+1}\lambda(M_n) + t^{n+1}\lambda(M_{n+1}) - t^{n+1}\lambda(L_{n+1}) = 0$$

missing some initial terms ↓ bring here

Sum as $n \to \infty$

$$t\,P(K,t) - t\,P(M,t) + P(M,t) - P(L,t) = \text{poly in } \mathbb{Z}[t]$$

$$(1-t)P(M,t) = P(L,t) - t\,P(K,t) + \text{poly}$$

$x_s K = 0$ (since K = kernel)

$x_s L = 0$ since L looks like $M/x_s M$

So K and L are A/x_sA-modules, generated by $x_1, \dots, x_{s-1}$ as A_0-algebra (since A was generated by $x_1, \dots, x_s$, sent x_s to 0), so $P(L,t) = \dfrac{\text{poly}}{(1-t)^{s-1}}$ $P(K,t) = \dfrac{\text{poly}}{(1-t)^{s-1}}$

$d(M)$ = order of pole of $P(M,t)$ at $t=1$

Cor. Assume $\deg x_i = 1$ for $1 \le i \le s$
Then $\exists$ poly $f_M(x) \in \mathbb{Q}[x]$ of degree $\underline{\underline{d(M)-1}}$ such that
$\lambda(M_n) = f_M(n)$ for sufficiently large n

$\lambda(M_n)$ = coeff of t^n in $P(M,t) = \dfrac{g(t)}{(1-t)^d}$
where $d = d(M)$

$$(1-t)^{-d} = \sum_{i=0}^{\infty} \binom{-d}{i} t^i = \sum_{i=0}^{\infty} (-1)^i \binom{d+i-1}{i} t^i$$

$$\binom{-d}{i} = \frac{(-d)(-d-1)\cdots(-d-i+1)}{i!}$$

$$g(t) = \sum_{j=0}^{e} a_j t^j$$

$\lambda(M_n) =$ coeff of t^n in

$$\sum_{i=0}^{\infty} \sum_{j=0}^{e} (-1)^i a_j t^j \binom{d+i-1}{i} t^i =$$

$$\sum_{n=0}^{\infty} \sum_{\substack{i,j \geq 0 \\ i+j=n \\ j \leq e}} (-1)^i a_j \binom{d+i-1}{i} t^n$$

$i+j=n$

Valid for $n \geq e$

$$\text{So} \quad \sum_{n=0}^{\infty} \sum_{j=0}^{e} (-1)^{n-j} a_j \binom{d+n-j-1}{n-j} t^n$$

$$\|$$

$$\lambda\cdot(M_n)$$

$$\lambda(M_n) = \sum_{j=0}^{e} (-1)^{n-j} a_j \binom{n+d-j-1}{d-1}$$

$$\lambda(M_n) = f_m(n) \swarrow \text{Hilbert polynomial}$$

degree $d-1$

Lecture 23 (2009-04-27)

4/27/09

A noeth graded ring, M f.g. graded A-module

λ function from Grothendieck group of f.g. A-modules to $\mathbb{Z}$ satisfying

$$0 \to M_1 \to M_2 \to M_3 \to 0 \quad \Rightarrow \quad \lambda(M_2) = \lambda(M_1) + \lambda(M_3)$$

Basic example: A_0 = field, $\lambda(M)$ = dimension

also: $\lambda(M)$ = length of M, a simple A_0-module

where length = length of composition series

$$M = M_n \supseteq M_{n-1} \supseteq \cdots \supseteq M_0 = 0 \quad \text{with } M_i/M_{i-1} \text{ simple}$$

M has length n, by Jordan-Holder indep. of M_i's

typically, A_0 is Artin local ring, $m \subset A_0$ is the maximal ideal, $m^n = 0$

only simple module is A/m

$$\begin{tikzcd} K \arrow[r, hook] \arrow[rr, bend right, "\text{could be isomorphism}"'] & A_0 \arrow[r, two heads] & A_0/m \end{tikzcd}$$

$A_0 = \mathbb{Z}/p^n\mathbb{Z}$ has length n, but doesn't have dimension n over anything

Each M_n is a f.g. A_0-module

$$P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n \in \mathbb{Z}[[t]].$$

Thm. $P(M,t)$ rational function of t, $P(M,t) = \dfrac{f(M,t)}{\prod_{i=1}^{s}(1-t^{k_i})}$ $\quad f \in \mathbb{Z}[t]$

where M generated by $x_1, \ldots, x_s$, of degree $\deg(x_i) = k_i$

look at order of pole of $P(M,t)$ at $t=1$; it is $\leq \sum_{i=1}^{s} k_i$ (could be cancellation)

Corollary. If all $k_i = 1$, (i.e. $A = A_0[A_1]$, i.e. all generators of A over A_0 are in A_1), for $n >> 0$, $\lambda(M_n)$ is a polynomial in n, with rational coefficients, of degree $d-1$ where d = order of pole

$\lambda(M_n)$ = Hilbert function of n

x non-zero divisor in M

$$0 \to M \xrightarrow{x} M \to L \to 0$$

ERASED

Now we work with A noeth local ring, M f.g. A-module
m = max ideal

Q = m-primary ideal $(\exists n : m \supseteq Q \supseteq m^n)$

(Q p-primary for prime p
if $xy \in Q \Rightarrow x \in p$ or $y^n \in Q$
for some n)

$\{M_n\}$ stable Q-filtration

Thm. (i). M/M_n is of finite length $\forall n > 0$

(ii) for $n >> 0$, $g(n) = \ell(M/M_n)$ is a polynomial in n of degree $\leq s$

s = # generators of Q

(iii) degree and leading coefficient of $g(n)$ depend only on M, Q, not on $\{M_n\}$.

Example $A = A_0[x_1, \ldots, x_s]$

A_n generated by monomials $x_1^{m_1} \cdots x_s^{m_s}$

$m_1 + \cdots + m_s = n$

# of these: $\binom{s+n-1}{s-1}$

$P(A,t) = (1-t)^{-s}$

Pf. (i) M/M_n is f.g. A-module, so f.g. A/Q^n module

$\ell(A/Q^n) < \infty \Rightarrow \ell(M/M^n) < \infty$

to prove $\ell(A/m^n)$ finite, reduces to proving $\ell(m^n/m^{n+1})$ finite, but this is finite dim vector space over A/m

(ii) $G_Q(A) = \bigoplus_{n=0}^{\infty} Q^n/Q^{n+1} = A/Q \oplus Q/Q^2 \oplus \cdots$

let $Q = (x_1, \ldots, x_s)$, so $\bar{x}_1, \ldots, \bar{x}_s$ generate $G_Q(A)$ as A/Q algebra

each $\bar{x}_i$ has deg 1

$\ell(M_n/M_{n+1}) = f(n)$ is polynomial in n of degree $\leq d-1 \leq s-1$

d = order of pole of $P(M,t) \leq \sum k_i = s$

$$\underset{\underset{\ell(M/M_{n+1})}{\uparrow}}{\ell_{n+1}} - \underset{\ell(M/M_n)}{\ell_n} = f(n)$$ for n large so ℓ_n eventually polynomial by "integrating"

Let $\bar{M}_n$ be another stable Q-filtration

$\tilde{g}(n) = \ell(M/\bar{M}_n)$

Two filtrations have bounded difference

So $g(n+n_0) \geq \tilde{g}(n)$
$\tilde{g}(n+n_0) \geq g(n)$

if different degree and/or different leading term, one would go to ∞ much faster than other

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Let χ_Q^M be the polynomial $\chi_Q^M(n) = \ell(M/Q^n M)$ $n >> 0$

Let $\chi_Q = \chi_Q^A$ (i.e. $M = A$)

Corollary $\ell(A/Q^n) =$ polynomial of degree $\le s = \#$ generators of Q

all χ_Q's have same degree, degree for $Q \geq$ degree for m

$$m \supseteq Q \supseteq m^r$$

$$m^n \supseteq Q^n \supseteq m^{rn}$$

$$\chi_m(n) \le \chi_Q(n) \le \chi_m(rn)$$

Let $d(A) =$ common degree of $\chi_Q(A)$ for all Q

Let $\delta(A) =$ minimal # of generators of an m-primary ideal Q

Thm. $d(A) = \delta(A) = \dim(A)$ for A noeth local

__Def__ A regular if $m = (x_1, \dots, x_s)$, $s = \dim(A)$

A regular $\Leftrightarrow$ $G_m(A) = A/m \oplus m/m^2 \oplus \cdots$ is a poly ring in s variables over A/m

Ex. $A = k[[x,y]]$

$$M = (y^2 - x^3)$$

$m = (x,y)$

$m^2 = (x^2, xy) \subseteq (x)$

$m^3 = (x^3, x^2y)$

$\ell(A/m^n)$	
1	$n=1$
3	$n=2$
5	$n=3$
7	$n=4$
$2n-1$	

$m \supseteq (x) \supseteq m^2$

To prove: $\delta(A) \geq d(A) \geq \dim(A) \geq \delta(A)$

Prop 11.8 in book M f.g. A-module, x non-zero-divisor in M

$$M' = M/xM$$

Claim: $\deg \chi_Q^{M'} \le \deg \chi_Q^M - 1$

$M \xrightarrow{\simeq} N$

Let $N = xM$ $\quad M \simeq N \quad 0 \to N \to M \to M' \to 0$

Let $N_n = N \cap Q^n M \quad M' = M/xM$

$$0 \to N/N_n \to M/Q^n M \to M'/Q^n M' \to 0$$

Let $g(n) = \ell(N/N_n)$

$g(n) = \chi_Q^M(n) + \chi_Q^{M'}(n)$

$N \simeq M$ Artin-Rees $\Rightarrow N_n$ stable Q-filtration on $N = M$

so g, χ_Q^M have same degree, leading coeff, so same leading term

subtract leading term from both sides, get what we wanted

Cor (11.9 in book) A noeth local ring, x non-zero-divisor

$d(A/x) \le d(A) - 1$

$(M=A)$ Next time: $d(A) \geq \dim(A)$

Lecture 24 (2009-05-05)

4/5/09

Regular local rings

Suppose A noeth local ring, $\dim(A) = d$
m maximal, $k = A/m$

Prop. TFAE:

$$\left.\begin{array}{l} 1)\ \dim_{A/m}(m/m^2) = d \\ 2)\ m \text{ generated by } d \text{ elements} \\ 3)\ G_m(A) \simeq \text{polynomial ring in } d \text{ variables} \end{array}\right\} \text{Def. } A \text{ is regular.}$$

Pf. 1) $\Rightarrow$ 2) by Nakayama

Clearly $\dim_{A/m}(m/m^2) \geq d$

2) $\Rightarrow$ $\dim_{A/m}(m/m^2) \leq d$ so

2) $\Rightarrow$ $\dim = d$

$$G_m(A) = A/m \oplus \overset{\dim d}{m/m^2} \oplus \cdots$$

3) $\Rightarrow$ 1)

m generated by d elements $x_1, \ldots, x_d$

Then $\bar{x}_1, \ldots, \bar{x}_d$ generate $G_m(A)$

$G_m(A)$ is a quotient of $k[\bar{x}_1, \ldots, \bar{x}_d]$

z is a non-zero divisor in graded ring R

$\dim(R/z) \leq \dim(R) - 1$

(Hilbert poly)

Suppose $G_m(A) = k[\bar{x}_1, \ldots, \bar{x}_d]/I$

$I \neq 0$, $\exists z \in I \neq 0$, z non-zero-div

Hilbert function $G_m(A) \leq$ Hilbert function $k[\bar{x}_1, \ldots, \bar{x}_d]/z$

degree $\leq d-1$

degree $= \dim(A)$

Prop. A ring I ideal, $\bigcap_n I^n = 0$

$G_I(A)$ integral domain $\Rightarrow$ A integral domain

Pf. Let $x, y \in A$, $x \neq 0$, $y \neq 0$

$\exists n : x \in I^n, x \notin I^{n+1}$ $\exists m$ $y \in I^m, y \notin I^{m+1}$

$\bar{x} \neq 0 \in I^n/I^{n+1}$ $\bar{y} \neq 0 \in I^m/I^{m+1}$

so

$\bar{x}\bar{y} \neq 0 \in I^{m+n}/I^{m+n+1}$

$\bar{x}\bar{y} \Rightarrow xy \neq 0$

so A regular local ring is an integral domain

Corollary. A noeth local, dim = 1 Then

A regular $\Leftrightarrow$ A DVR $\Leftrightarrow$ A integrally closed

Prop $G_m(A)$ integrally closed $\Rightarrow$ A integrally closed

Cor. A regular $\Rightarrow$ A integrally closed

$A = \dfrac{k[x,y,z,w]}{(xy - zw)}$ localize at 0 to get local ring

not regular has dim 3 maximal ideal generated by 4 elements

Exercise (?) This is integrally closed, not UFD

Cor. A noeth local

A regular $\Rightarrow$ $\hat{A}$ regular

$\dim(A) = \dim(\hat{A})$

$m_A/m_A^2 \simeq m_{\hat{A}}/m_{\hat{A}}^2$

A regular $\Leftrightarrow$ $\dim(A) = \dim(m_A/m_A^2)$

Suppose $k' \hookrightarrow \hat{A}$ k' any field

$$\begin{tikzcd} k' \arrow[r, hook] \arrow[rr, bend right, "\text{iso}"'] & \hat{A} \arrow[r] & \hat{A}/m = k \end{tikzcd}$$

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$\hat{A}$ complete noeth local regular ring, dim $= d$

then $\hat{A} \simeq k[[t_1, \ldots, t_d]]$

Can take $t_1, \ldots, t_d$ to be any set of generators for m_A

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend left, "\sim"] & A \arrow[r] & k \arrow[d, phantom, "\shortparallel"] \\ & & A/m \end{tikzcd}$$

equivalent to $A \simeq m \oplus k$

$\mathrm{Der}_k(A, k) \simeq \mathrm{Hom}_A(m, k) \simeq \mathrm{Hom}_{A/m}(m/m^2, k)$

$D \in \mathrm{Der}_k(A, k)$ $\qquad mk = 0$

Let $D/_m = \lambda$ $\qquad \lambda(ax) \overset{x \in m}{=} a\lambda(x) + \underset{0}{\underset{\shortparallel}{x\lambda(a)}}$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \\ M/mM \arrow[ur] & \end{tikzcd}$$

m kills N

Given $\lambda \in \mathrm{Hom}_A(m, k)$

define D by $D(x, y) = \lambda(x)$

$x \in m$, $y \in k$ $\qquad \mathrm{Der}_k(A, k) = \mathrm{Hom}_A(\Omega_{A/k}, k)$

For A domain, $K =$ fraction field of A, choose k perfect field

then A regular $\Leftrightarrow$ $\Omega_{A/k}$ is free of rank d

$\Omega_{A/k} \underset{A}{\otimes} K = \Omega_{K/k}$

$\dim(\Omega_{K/k}) = \mathrm{tr}\deg(K/k) \underset{\dim(A)}{\shortparallel}$

A regular $\Rightarrow$ $\mathrm{Hom}(\Omega_{A/k}, k)$ has $\dim(d)$

$\Updownarrow$

$\mathrm{Hom}(\Omega_{A/k} / m\Omega_{A/k}, k)$ has $\dim(d)$ dual space?

so $\dim(\Omega_{A/k} / m\Omega_{A/k}) = d$

so $\Omega_{A/k}/m\Omega_{A/k}$ has d generators

$\to$ let M be kernel in

$0 \to M \to A^d \to \Omega_{A/k} \to 0$

$0 \to M \otimes K \to K^d \to \underset{\text{erased}}{} \to 0$

$M \otimes K$ exact

So M torsion, but $M \subseteq A^d$, so

$M = 0$

So $\Omega_{A/k} = A^d$

$\mathbb{A}^2$ $k[x,y]$ k alg closed

$f(x,y)=0$

$\Omega_{k[x,y]/k}$ = free module on dx, dy

$B = k[x,y]/f(x,y)$

$\Omega_{B/k}$ = free module on $dx, dy / df$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ when free module of rank 1?

iff eith $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ is a unit in local ring B_P

$\Longleftrightarrow$ either $\frac{\partial f}{\partial x}(P) \neq 0$ or $\frac{\partial f}{\partial y}(P) \neq 0$

Singular points $\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$ $f = 0$

Example; affine elliptic curves, char $\neq 2, 3$

$y^2 = x^3 + Ax + B$

A $k[x_1, \ldots, x_n]_M$ M maximal ideal

$k = \bar{k}$ $M \Longleftrightarrow$ points P

P is non-singular $\Longleftrightarrow$ A regular (definition)