Math 317 - Algebraic Topology

Lectures by Benson Farb
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The University of Chicago, Fall 2012

Introduction

Math 317 is one of the nine courses offered for first-year mathematics graduate students at the University of Chicago. It is the first of three courses in the year-long geometry/topology sequence.

These notes were live-TeXed, though I edited for typos and added diagrams requiring the TikZ package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to chonoles@math.uchicago.edu.

Acknowledgments

Thank you to all of my fellow students who sent me corrections, and who lent me their own notes from days I was absent. My notes are much improved due to your help.

Lecture 1 (2012-10-01)

Today I'll try to present a general idea of algebraic topology.

Definition. A category $\mathcal{C}$ is a set $\mathcal{C}$ (though not always really a set) of objects and a set

$$\mathrm{Mor}(\mathcal{C}) = \{f : A \to B \mid A, B \in \mathcal{C}\}$$

of morphisms between the objects of $\mathcal{C}$. We can always compose morphisms (whose domains and codomains are appropriately related), this composition is always associative, and for all objects $A \in \mathcal{C}$, there is a morphism $\mathrm{id}_A : A \to A$ that acts as the identity for composition.

Examples. The following are some common examples of categories.

- $\mathcal{C} = \{\text{topological spaces}\}$, and $\mathrm{Mor}(\mathcal{C}) = \{\text{continuous maps}\}$
- $\mathcal{C} = \{\text{topological spaces}\}$, and $\mathrm{Mor}(\mathcal{C}) = \{\text{homeomorphisms}\}$
- $\mathcal{C} = \{\text{vector spaces over a field } K\}$, and $\mathrm{Mor}(\mathcal{C}) = \{K\text{-linear maps}\}$
- $\mathcal{C} = \{\text{abelian groups}\}$, and $\mathrm{Mor}(\mathcal{C}) = \{\text{homomorphisms}\}$

Definition. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ is a mapping that takes objects $A \in \mathcal{C}$ to objects $F(A) \in \mathcal{D}$, and morphisms $(f : A \to B)$ between objects of $\mathcal{C}$ to morphisms $(F(f) : F(A) \to F(B))$ between the corresponding objects of $\mathcal{D}$.

We'll often refer to $F(f)$ as f_* when the functor F is understood.

We can also define a contravariant functor to be a functor that reverses the direction of morphisms, so that $F(f) : F(B) \to F(A)$. In this situation, the shorthand for $F(f)$ will be f^*.

In either case, it must satisfy $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. It also must satisfy functoriality; that is, we must have $F(f \circ g) = F(f) \circ F(g)$, or in shorthand $(f \circ g)_* = f_* \circ g_*$. This goes in the other order for contravariant functors.

Examples. The following are some common examples of functors.

- $\mathcal{C} = \{\text{finite sets}\}$ with $\mathrm{Mor}(\mathcal{C}) = \{\text{set maps}\}$, $\mathcal{D} = \{\mathbb{R}\text{-vector spaces}\}$ with $\mathrm{Mor}(\mathcal{D}) = \{\text{linear maps}\}$, and $F : \mathcal{C} \to \mathcal{D}$ defined on objects by

  $$S \mapsto \text{free } \mathbb{R}\text{-vector space on } S$$

  and for a morphism $f : S \to T$, $F(f)$ is the linear map uniquely determined by sending the basis element s to the basis element $f(s)$.
- $\mathcal{C} = \{\mathbb{R}\text{-vector spaces}\}$ with $\mathrm{Mor}(\mathcal{C}) = \{\text{linear maps}\}$, and $F : \mathcal{C} \to \mathcal{C}$ defined on objects by $V \mapsto V^*$, and on morphisms by $(A : V \to W) \mapsto (A^* : W^* \to V^*)$.

Broadly speaking,

$$\text{algebraic topology} \quad \simeq \quad \text{construction of functors } \{\text{spaces}\} \to \{\text{algebraic objects}\}.$$

The functors we'll look at in this class are H_i, H^i, π_1, and π_n.

Here are some demonstrations of the power of functors. Let's suppose we know there exists a functor $F_i : \{\text{spaces}\} \to \{\text{abelian groups}\}$ such that $F_i(\mathbb{D}^n) = 0$ for any n, $F_i(\mathbb{S}^n) = 0$ for $n \neq i$, and $F_i(\mathbb{S}^i) \neq 0$. We don't have to know what else it does.

Example. From this assumption alone, we can prove that $\mathbb{S}^n \cong \mathbb{S}^m \iff n = m$, and as a corollary, prove "invariance of dimension", i.e. $\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$.

Proof. Suppose $h : \mathbb{S}^n \to \mathbb{S}^m$ is a homeomorphism, with inverse g. Then applying the functor F_n,

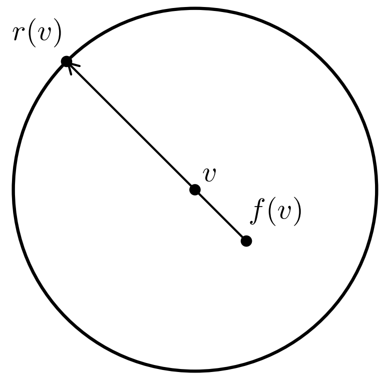
$$g \circ h = \mathrm{id}_{\mathbb{S}^n} \implies g_* \circ h_* = \mathrm{id}_{F_n(\mathbb{S}^n)},$$

$$h \circ g = \mathrm{id}_{\mathbb{S}^m} \implies h_* \circ g_* = \mathrm{id}_{F_n(\mathbb{S}^m)}.$$

This implies that $h_* : F_n(\mathbb{S}^n) \to F_n(\mathbb{S}^m)$ is a bijection, and hence an isomorphism. But $F_n(\mathbb{S}^n) \neq 0$ and $F_n(\mathbb{S}^m) = 0$ unless $n = m$. □

Example (Brouwer Fixed Point Theorem)**.** Any continuous $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point ($n \geq 2$).

Proof. Suppose there are no fixed points. Then we can define $r : \mathbb{D}^n \to \partial\mathbb{D}^n = \mathbb{S}^{n-1}$ by drawing a line from $f(v)$ to v (this is possible only because we are assuming they are different) and extending it until it hits the boundary of $\mathbb{D}^n$, and setting that to be $r(v)$.



You can prove that this is continuous on your own. Note that for any $v \in \mathbb{S}^{n-1}$, we will have that $r(v) = v$. Thus, letting $i : \mathbb{S}^{n-1} \to \mathbb{D}^n$ be the obvious inclusion map, we have that $r \circ i = \mathrm{id}_{\mathbb{S}^{n-1}}$ (we say that "r is a retraction of $\mathbb{D}^n$ onto $\mathbb{S}^{n-1}$"). Because $r \circ i = \mathrm{id}_{\mathbb{S}^{n-1}}$, we have that

$$F_{n-1}(r) \circ F_{n-1}(i) = \mathrm{id}_{F_{n-1}(\mathbb{S}^{n-1})}$$

so that $F_{n-1}(r)$ must be surjective. But $F_{n-1}(r) : F_{n-1}(\mathbb{D}^n) \to F_{n-1}(\mathbb{S}^{n-1})$ can't be surjective because $F_{n-1}(\mathbb{D}^n) = 0$ and $F_{n-1}(\mathbb{S}^{n-1}) \neq 0$. This is a contradiction. □

A final comment: a good method for turning invariants for spaces into invariants for maps is to associate to a map $f : X \to Y$ its graph $\Gamma_f \subset X \times Y$.

Lecture 2 (2012-10-03)

There are some corrections for the homework: on problem 2a, it should read

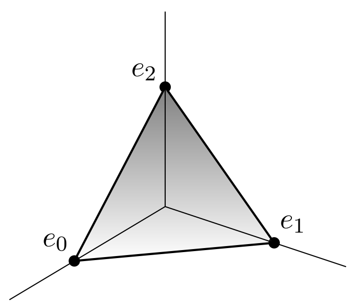
$$H_0(\mathcal{C}) = \widetilde{H}_0(\mathcal{C}) \oplus \mathbb{Z}$$

and on problem 3, you should show that the following diagram commutes:

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "{[v_0 \cdots v_n]}"'] & \Delta^m \arrow[r, "\tau"] & Y \\
X \arrow[urr, "f"'] & &
\end{tikzcd}$$

Simplices and Δ-complexes

Definition. The standard (ordered) n-simplex $[v_0 \cdots v_n]$ is the convex hull of the set $\{e_1, \ldots, e_{n+1}\}$ where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{n+1}$. For example, when $n = 2$,



Equivalently, it is the set

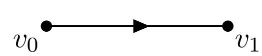
$$\left\{ \sum_{i=1}^n a_i e_i \in \mathbb{R}^{n+1} \mid a_i \geq 0, \sum_{i=1}^n a_i = 1 \right\}.$$

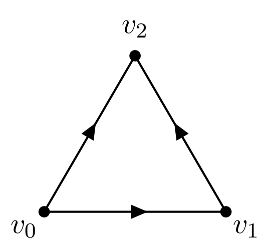
We will often want to specify using coordinates. Let $v_0 = 0$, and $v_i = e_i$ for $i = 1, \ldots, n+1$. Then

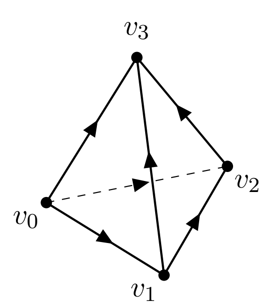
$$\Delta^n = \text{ convex hull } (\{v_1 - v_0, \ldots, v_{n+1} - v_0\}).$$

For example,

$\Delta^0 \quad =$ 

$\Delta^1 \quad =$ 

$\Delta^2 \quad =$ 

$\Delta^3 \quad =$ 

Note that an ordered n-simplex has $n+1$ "face maps"; for each $i = 0, \ldots, n$,

$$[v_0 \cdots \widehat{v_i} \cdots v_n] = \text{ an ordered } (n-1)\text{-simplex ("subsimplex of } [v_0 \cdots v_n]\text{")}.$$

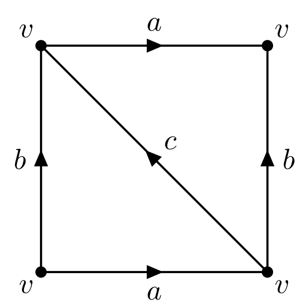
For example, for $n = 2$, we get canonical linear maps

$$\begin{tikzcd}
 & & {[v_2]} \\
 & {[v_1v_2]} \arrow[ur] \arrow[dr] & \\
 & & {[v_1]} \\
 & & {[v_2]} \\
{[v_0v_1v_2]} \arrow[uuur] \arrow[r] \arrow[dddr] & {[v_0v_2]} \arrow[ur] \arrow[dr] & \\
 & & {[v_0]} \\
 & & {[v_1]} \\
 & {[v_0v_1]} \arrow[ur] \arrow[dr] & \\
 & & {[v_0]}
\end{tikzcd}$$

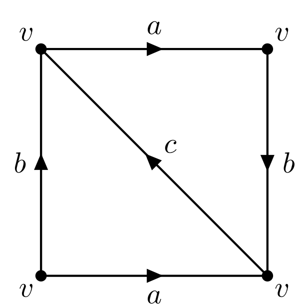
Definition. Let X be a topological space. A Δ-complex structure on X is a decomposition of X into simplices. Specifically, it is a finite collection $S = \{\Delta_i\}$ of simplices with continuous maps that are injective on their interiors, that also satisfies

- $\bigcup_i \sigma_i(\Delta_i) = X$,
- For all $x \in X$, there is a unique i such that $x \in \text{Int}(\sigma_i(\Delta_i))$.
- If $\sigma : \Delta \to X$ is an element of S, then all subsimplices τ of σ are also elements of S. In other words, S is closed under taking faces.

Example. The torus $\mathbb{T}^2$ can be given a Δ-complex structure as follows:

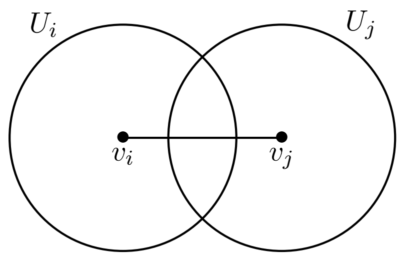


Example. The Klein bottle can be given a Δ-complex structure as follows:

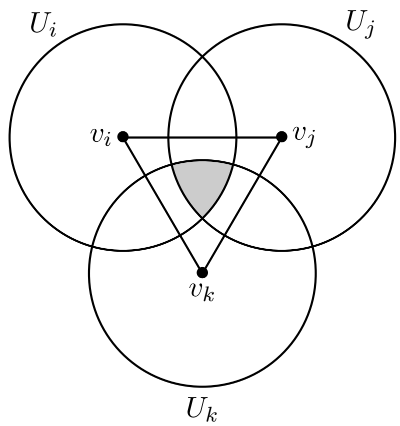


Example. Let X be a topological space, and let $\{U_i\}$ be an open cover of X (assume indexed by an ordered set, induces order for simplices). The nerve of the cover is the Δ-complex specified by

- A vertex v_i for each U_i
- If $U_i \cap U_j \neq \varnothing$, the 1-simplex $[v_i v_j]$



- If $U_i \cap U_j \cap U_k \neq \varnothing$, the 2-simplex $[v_i v_j v_k]$



Simplicial Homology

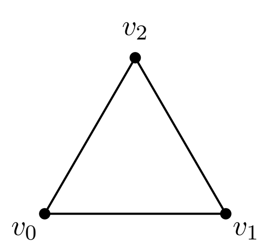
We want, for any Δ-complex X, and $i \geq 0$, an abelian group $H_i(X)$. We want this to be

1. **Computable**
2. Topologically invariant: if X, Y are Δ-complexes and $X \cong Y$, then $H_i(X) \cong H_i(Y)$.
3. Non-triviality: $H_n(\mathbb{S}^n) \neq 0$.

Step 1 (topology): We input a Δ-complex structure on X. We output, for each $n \geq 0$,

$$C_n(X) = \text{free abelian group on } \{(\text{ordered}) \ n\text{-simplices in } X\}.$$

Example. Let X be a 2-simplex,

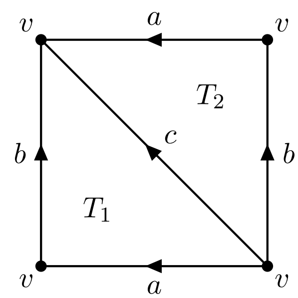


Then we have

- $C_0(X) = \mathbb{Z}^3 = \langle [v_0], [v_1], [v_2] \rangle$
- $C_1(X) = \mathbb{Z}^3 = \langle [v_0 v_1], [v_0 v_2], [v_1 v_2] \rangle$
- $C_2(X) = \mathbb{Z} = \langle [v_0 v_1 v_2] \rangle$

- $C_n(X) = 0$ for $n \geq 3$

Example. Let X be the torus,



Then we have

- $C_0(X) = \mathbb{Z} = \langle [v] \rangle$
- $C_1(X) = \mathbb{Z}^3 = \langle a, b, c \rangle$
- $C_2(X) = \mathbb{Z}^2 = \langle T_1, T_2 \rangle$
- $C_n(X) = 0$ for $n \geq 3$

Note that we are abusing notation and identifying simplices with their maps to the space.

Step 2 (algebra): Because $C_n(X)$ is the free abelian group on the simplices, a $\tau \in C_n(X)$ is uniquely expressed as

$$\tau = \sum_{i=1}^{r} a_i \sigma_i, \; a_i \in \mathbb{Z}.$$

For all $n \geq 1$, we define homomorphisms $\delta_n : C_n(X) \to C_{n-1}(X)$ by specifying them on the generators of $C_n(X)$: for any n-simplex $\sigma : \Delta^n \to X$, we let

$$\delta_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 \cdots \widehat{v_i} \cdots v_n]}.$$

The key property of these homomorphisms is that $\delta_n \circ \delta_{n-1} = 0$. Note that we also needed the ordering to define this.

We will compute some homology next class.

Lecture 3 (2012-10-05)

Let X be a Δ-complex. This is essentially a simplicial complex with extra structure, namely, an ordering of the vertices compatible with σ-maps.

For each n,

$$\Delta_n(X) = C_n(X) = \text{free abelian group on the set of ``allowable'' maps } \Delta^n \to X.$$

For each n-simplex in X (i.e. $\Delta^n \xrightarrow{e_\alpha} X$), there are several Δ-maps

$$\begin{tikzcd} \Delta^n \arrow[r, "e_\alpha"] & X \\ \Delta^{n-1} \arrow[u] \arrow[ur, "e_\beta"'] & \end{tikzcd} \qquad \text{(these are the faces of } e_\alpha\text{)}$$

Given $e_\alpha(v_0, \dots, v_n)$, we define

$$\partial e_\alpha = \sum_{i=0}^{n} (-1)^i e_\alpha|_{(v_0, \dots, \widehat{v_i}, \dots, v_n)}.$$

The key property of this construction is that $\partial \circ \partial = 0$.

Now we do some pure algebra.

Definition. A chain complex is a collection of abelian groups C_i and homomorphisms $\partial_i : C_i \to C_{i-1}$ such that $\partial_{i-1} \circ \partial_i = 0$. We write this as

$$\cdots \longrightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

We say that $\alpha \in C_n$ is a cycle if $\partial_n(\alpha) = 0$ and $\beta \in C_n$ is a boundary if there is some $\gamma \in C_{n+1}$ such that $\partial_{n+1}(\gamma) = \beta$. In other words,

$$n\text{-cycles} = \ker(\partial_n), \quad n\text{-boundaries} = \operatorname{im}(\partial_{n+1}).$$

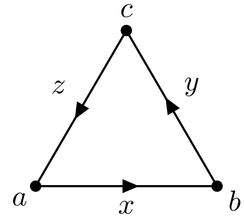
All boundaries are cycles, but not all cycles are boundaries. Letting Z_n be the subgroup of cycles in C_n, and B_n the subgroup of boundaries in C_n, we have $B_n \subseteq Z_n$, and we define

$$H_n = Z_n / B_n.$$

Applying this to topology, we define

$$H_n(X) = n\text{th homology group of } (\Delta_n(X), \partial_n).$$

Example. Consider the triangle



We have that

$$\Delta_0 = \langle a, b, c \rangle \cong \mathbb{Z}^3, \quad \Delta_1 = \langle x, y, z \rangle \cong \mathbb{Z}^3.$$

The map ∂_1 sends

$$x \longmapsto b - a, \quad y \longmapsto c - b, \quad z \longmapsto c - a$$

or expressed as a matrix,

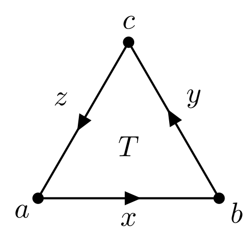
$$\partial_1 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Thus,

$$\ker(\partial_1) = \langle x + y - z \rangle \cong \mathbb{Z}.$$

Of course, we have $\ker(\partial_0) = \mathbb{Z}^3$, and $\operatorname{im}(\partial_1)$ is the set of things which sum to zero, so $H_0 \cong \mathbb{Z}$.

Example. Consider the 2-simplex



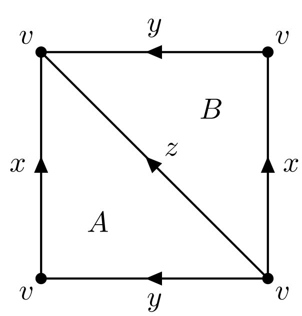
We have that

$$\Delta_0 = \langle a, b, c \rangle \cong \mathbb{Z}^3, \quad \Delta_1 = \langle x, y, z \rangle \cong \mathbb{Z}^3, \quad \Delta_2 = \langle T \rangle.$$

The map ∂_1 is the same as before, and $\partial_2(T) = y - z + x$. In this case we get

$$H_0 \cong \mathbb{Z}, \quad H_1 \cong 0, \quad H_2 \cong 0.$$

Example. Consider the torus



We have that

$$\Delta_0 = \langle v \rangle, \qquad \Delta_1 = \langle x, y, z \rangle, \qquad \Delta_2 = \langle A, B \rangle$$

with $\partial_0 = 0$, $\partial_1 = 0$, and

$$\partial_2(A) = y + z - x, \quad \partial_2(B) = y - x + z$$

which implies that $\ker(\partial_2) = \langle A - B \rangle$ and $\operatorname{im}(\partial_2) = \langle y + z - x \rangle$. Thus,

$$H_1 = \langle x, y, z \rangle / \langle y + z - x \rangle \cong \mathbb{Z}^2, \quad H_2 = \langle A, B \rangle / \langle A - B \rangle \cong \mathbb{Z}.$$

Lecture 4 (2012-10-08)

Homotopy was invented to study the topological invariance of homology groups.

Proposition. *Let X be a Δ-complex with path components $X_1, \ldots, X_r$. Then $H_0(X) \cong \mathbb{Z}^r$ and furthermore, for all $i \geq 0$,*

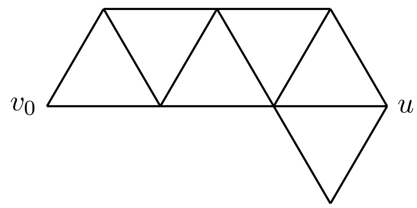
$$H_i(X) = \bigoplus_{j=1}^{d} H_i(X_j).$$

Proof. We'll show that X connected implies that $H_0(X) \cong \mathbb{Z}$. Then we just have to prove the general part.

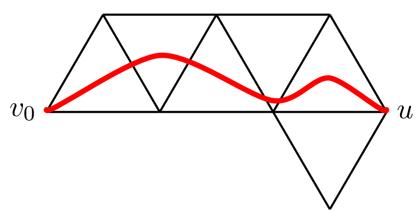
Note that

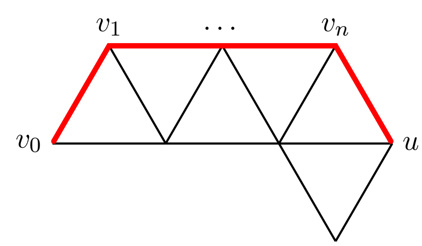
$$H_0(X) = Z_0(X)/B_0(X) = C_0(X)/\operatorname{im}(\partial_1).$$

Pick a vertex v_0, and let u be any other vertex.



We can see that $[u] = [v_0] \in H_0(X)$ because X is path-connected, so there exists a path from v_0 to u, which we can homotope to a path of edges $[v_0v_1] \cup [v_1v_2] \cup \cdots [v_nu]$.





Let

$$\sigma = [v_0v_1] + [v_1v_2] + \cdots + [v_nu],$$

so that

$$\partial\sigma = (v_1 - v_0) + (v_2 - v_1) + \cdots + (u - v_n).$$

Thus $u - v_0 = \partial\sigma \in \operatorname{im}(\partial_1)$, and thus $[u] = [v_0] \in H_0(X)$.

For any $w = \sum a_i v_i$, we therefore have that $[w] = \sum a_i[v_i] = \left(\sum a_i\right)[v_0]$. Thus, $[v_0]$ generates $H_0(X)$.

To prove that $[v_0]$ has infinite order in $H_0(X)$, we'll use contradiction. Suppose it had order n; then $n[v_0] = 0 \in H_0(X)$, so that there is some $\sigma \in C_1(X)$ such that $\partial\sigma = n[v_0]$. Now we define a homomorphism $\psi : C_0(X) \to \mathbb{Z}$ by sending every generator of $C_0(X)$ to 1. Thus, for any 1-simplex $[uv]$,

$$\psi(\partial[uv]) = \psi(v - u) = \psi(v) - \psi(u) = 0$$

but $n = n\psi(v_0) = \psi(nv_0) = \psi(\partial\sigma) = 0$, which is a contradiction.

Thus, we've shown that $H_0(X) \cong \mathbb{Z}$ for X connected. Now we want to prove that

$$H_i(X) = \bigoplus_{j=1}^{d} H_i(X_j).$$

Let

$$\phi_n : \bigoplus_{i=1}^{d} C_n(X_i) \to C_n(X)$$

be defined by sending σ to σ (since $X_i \subseteq X$, a chain on X_i is literally a chain on X) and we can get an inverse map by noting that, for any simplex σ, its image $\sigma(\Delta^n)$ is connected, and therefore can only lie in one connected component. Thus ϕ_n is an isomorphism.

Note that "ϕ commutes with ∂", i.e.

$$\phi \circ \partial_n^+ = \partial_n \circ \phi$$

where $\partial_n^+ = \oplus \partial_n^{X_i}$ (this fact is clear).

Now we will need a homological algebra fact. For any chain complexes $\mathcal{C} = \{C_n, \partial_n\}$ and $\mathcal{C}' = \{C'_n, \partial'_n\}$, a chain map $f : \mathcal{C} \to \mathcal{C}'$ is a collection of maps $f_n : C_n \to C'_n$ such that $f_n \circ \partial_n = \partial'_n \circ f_n$, or as a diagram,

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \end{tikzcd}$$

The homological algebra fact is that any chain map $f : \mathcal{C} \to \mathcal{C}'$ **functorially** induces a homomorphism $f_* : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$. Here is a proof.

Given a $\sigma \in Z_n(\mathcal{C})$, or equivalently $\partial_n \sigma = 0$, we get that

$$\partial_n(f_n \circ \sigma) = f_n(\partial_n \sigma) = f_n(0) = 0$$

so $f_n(\sigma) \in Z_n(\mathcal{C}')$.

We map $Z_n(\mathcal{C})$ to $H_n(\mathcal{C})$ by sending σ to $[\sigma]$, and likewise for $\mathcal{C}'$. To show f induces a compatible map on homology (the dashed arrow)

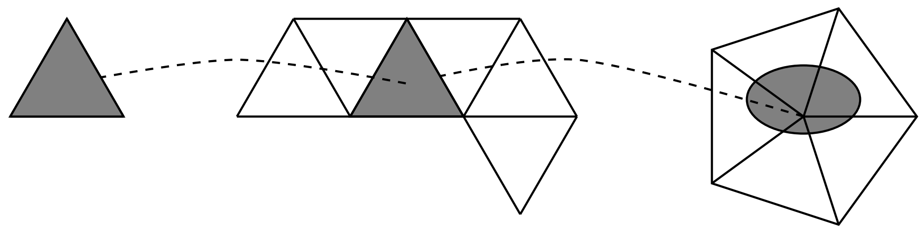
$$\begin{tikzcd} Z_n(\mathcal{C}) \arrow[r] \arrow[d] & Z_n(\mathcal{C}') \arrow[d] \\ H_n(\mathcal{C}) = Z_n(\mathcal{C})/B_n(\mathcal{C}) \arrow[r, dashed] & H_n(\mathcal{C}') = Z_n(\mathcal{C}')/B_n(\mathcal{C}') \end{tikzcd}$$

note that for any $\tau \in B_n(\mathcal{C})$, we have $\partial'(f(\tau)) = f(\partial\tau) = f(0) = 0$, so that $f(\tau) \in B_n(\mathcal{C}')$. Thus f_* is well-defined.

Now, to finish our proof, use the functoriality of the above map to see that, because ϕ is an isomorphism, it induces an isomorphism on homology. $\square$

Our next topic is the topological invariance of $H_n(X)$. Our goal is to show that if X and Y are Δ-complexes and $f : X \to Y$ is a homeomorphism, then $H_i(X) \cong H_i(Y)$ for all i.

The main problem is that if $\Delta_n \xrightarrow{\sigma} X$ is a simplex of X, then the composition $\Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y$ may not be a simplex of Y.



There are two main strategies for overcoming this. We can either work out the simplicial approximation theorem, in which we show that we can homotope our map sufficiently and subdivide our simplices sufficiently so that any map can be realized as a simplical map, or we can come up with a homology theory that we can't compute but know is isomorphic to what we want.

Lecture 5 (2012-10-10)

So far, we've seen how to start from a Δ-complex X and then form a complex of simplicial chains, and from there we use homological algebra to define the homology groups $H_i(x)$. But we'd like to be able to start from a topological space without a specified Δ-complex structure and be guaranteed to get isomorphic homology groups. In other words, we want to show that

$$X \cong Y \implies H_i(X) \cong H_i(Y) \text{ for all } i \geq 0.$$

As is common in mathematics, this becomes easier if we enlarge our category, in this case to the homotopy category.

Definition. Let X, Y be topological spaces, and $f, g : X \to Y$ maps. A homotopy from f to g, denoted $f \sim_F g$, is a map $F : X \times [0,1] \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$.

For $t \in [0,1]$, we write $F_t : X \to Y$ for the function $F_t(x) = F(x,t)$, so that $F_0 = f$ and $F_1 = g$.

Definition. A map $f : X \to Y$ is a homotopy equivalence if there is some $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. We say that $X \simeq Y$.

Remark. $f \sim g$ is an equivalence relation, and $X \simeq Y$ is an equivalence relation.

Definition. X is contractible if there is a point $x \in X$ such that $i : \{x\} \hookrightarrow X$ is a homotopy equivalence.

Example. Homotopy equivalence can't tell you anything about contractible spaces, not even their dimension, because $\mathbb{D}^n$ is contractible: the inclusion $\{0\} \hookrightarrow \mathbb{D}^n$ has as a homotopy inverse the map sending everything to 0. The homotopy is $F_t(v) = tv$.

Example. Any homeomorphism $f : X \to Y$ is also a homotopy equivalence.

Proposition. *Suppose H_i is any functor from*

$$\begin{Bmatrix} \textit{topological spaces (or } \Delta\textit{-complexes)} \\ \textit{with continuous maps} \end{Bmatrix} \to \begin{Bmatrix} \textit{abelian groups} \\ \textit{with homomorphisms} \end{Bmatrix}$$

satisfying

1. *Functoriality:* $(\mathrm{id}_X)_* = \mathrm{id}_{H_i(X)}$, $(f \circ g)_* = f_* \circ g_*$.
2. *Homotopy functor: if $f \sim g$, then $f_* = g_*$.*

Then H_* is a topological invariant; in fact, if $f : X \xrightarrow{\simeq} Y$, then $f_* : H_i(X) \xrightarrow{\cong} H_i(Y)$.

Proof. If $f : X \xrightarrow{\simeq} Y$ has $g : Y \to X$ as a homotopy inverse, then the fact that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$ implies that

$$f_* \circ g_* = \mathrm{id} \implies f_* \text{ surjective}, \qquad g_* \circ f_8 = \mathrm{id} \implies f_* \text{ injective},$$

and hence f_* is an isomorphism. ☐

How to Prove that H_i is a Homotopy Functor

Method 1: The Simplicial Approximation Theorem

The idea is that, given Δ-complexes X and Y with $f : X \xrightarrow{\cong} Y$, Step 1: We show there exists a simplicial homeomorphism $f' : X' \to Y$ (where X' is the barycentric subdivision of X)

Step 2: Then show that $H_i(X) \cong H_i(X')$.

Step 3: Then show that if f'' is another such map, then $(f'')|_* = (f')_*$.

Step 4: Now declare $f_* = f'_*$.

Method 2: Singular Homology

Step 1: Construct singular homology, H_i^S, which is easy to show is a homotopy functor.

Step 2: For any Δ-complex X, $H_i^S(X) \cong H_i(X)$.

We'll do it this way.

Singular Homology

Let X be any topological space, and for any $n \geq 0$, define

$$C_n(X) = \text{free abelian group on } \{\text{continuous maps } \Delta^n \to X\}.$$

Define $\partial_n : C_n(X) \to C_{n-1}(X)$ as the unique linear extension of

$$(\sigma : [v_0 \cdots v_n] = \Delta^n \to X) \mapsto \partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 \cdots \widehat{v_i} \cdots v_n]}$$

Check that $\partial_{n-1} \circ \partial_n = 0$. This is the complex of singular chains.

Definition. The singular homology $H_i^S(X)$ of a space X is

$$H_i^S(X) = H_i(\{C_n(X), \partial_n\}).$$

Induced maps: if $f : X \to Y$ is continuous, then we can send $\sigma : \Delta^n \to X$ to $(f \circ \sigma) : \Delta^n \to Y$, and then extend this to $f_\# : C_n(X) \to C_n(Y)$.

It is trivial to check that $f_\#$ is a chain map, and so induces $f_* : H_n^S(X) \to H_n^S(Y)$, and also that $(f \circ g)_\# = f_\# \circ g_\#$, which then implies that $(f \circ g)_* = f_* \circ g_*$.

Note that we're building a dictionary:

$\{C_n(X), \partial_n\}$	$\mathcal{C} = \{C_n, \partial_n\}$
continuous maps $f : X \to Y$	chain maps $f : \mathcal{C} \to \mathcal{D}$
$f \sim g$ homotopy	chain homotopy

Lecture 6 (2012-10-12)

Remember, this is our goal:

Theorem. *If $\mathcal{F} : \{\substack{\text{topological spaces} \\ \text{continuous maps}}\} \to \{\substack{\text{abelian groups} \\ \text{homomorphisms}}\}$ is a functor that is a homotopy functor, i.e.*

$$f \sim g \implies f_* = g_*,$$

then $h : X \xrightarrow{\simeq} Y$ (i.e. X and Y are homotopy equivalent) implies that $h_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ is an isomorphism.

Given a space X, we defined its singular chain complex $C_n^S(X)$ to be the free abelian group on the set $\{\text{continuous } \Delta^n \to X\}$, and then defined the singular homology $H_i^S(X)$ to be the homology of this chain complex. That this is functorial is trivial; the map $f_\# : C_n^S(X) \to C_n^S(Y)$ induced by an $f : X \to Y$ defined by $f_\#(\sum a_i\sigma_i) = \sum a_i(f \circ \sigma_i)$ is a chain map.

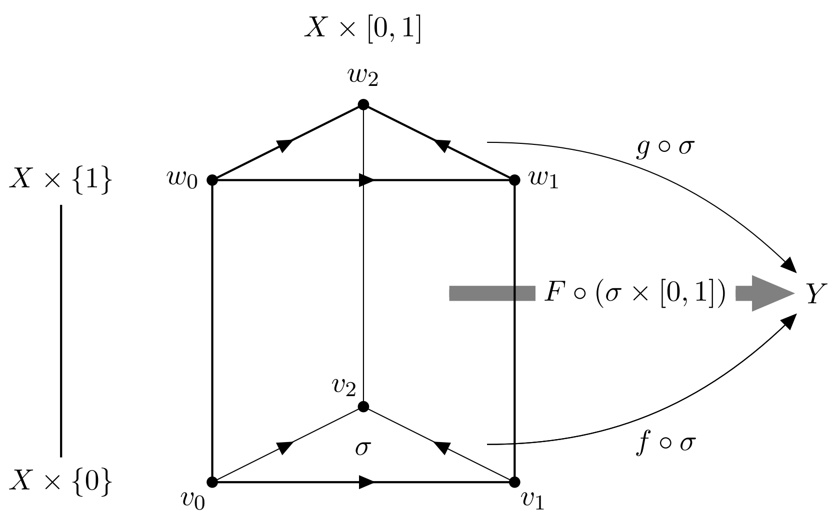
Proposition. *Given $f, g : X \to Y$ with $f \sim g$, then $f_* = g_* : H_i^S(X) \to H_i^S(Y)$ for all $i \geq 0$.*

Corollary. *A homotopy equivalence $h : X \xrightarrow{\simeq} Y$ induces an isomorphism $h_* : H_i^S(X) \xrightarrow{\cong} H_i^S(Y)$.*

Proof of Prop. Given a homotopy $F : X \times [0,1] \to Y$ with $F_0 = f$ and $F_1 = g$, we will construct a chain map $P_i : C_i^S(X) \to C_{i+1}^S(Y)$,

$$\begin{tikzcd}
C_{i+1}^S(X) \arrow[r, "\delta"] & C_i^S(X) \arrow[r, "\delta"] \arrow[dl, "P_i"] & C_{i-1}^S(X) \arrow[r] \arrow[dl, "P_{i-1}"] & \cdots \\
C_{i+1}^S(Y) \arrow[r, "\delta"] & C_i^S(Y) \arrow[r, "\delta"] & C_{i-1}^S(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Morally, P is the (unique linear extension of) the "prism operator" which takes a simplex $\sigma \in C_i(X)$ to a "prism" P which is then sent to Y by the homotopy F; thus, the prism allows us to compare $f \circ \sigma$ with $g \circ \sigma$. Of course, a prism is not a simplex, but we can view its image in Y as a chain in $C_{i+1}(Y)$ by appropriately subdividing it.



Taking the boundary of the prism operator produces

$$\partial P = \overbrace{g_\#}^{\text{top}} - \overbrace{f_\#}^{\text{bottom}} - \overbrace{P\partial}^{\text{sides}}.$$

A subdivision that works: given $\sigma : [v_0 \cdots v_i] \to X$, define $P\sigma$ by

$$P\sigma[v_0 \cdots v_{i+1}] = \sum_{j=0}^{i} (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0 \cdots v_j w_j \cdots w_i]}$$

Homological algebra interlude

Definition. Let $\phi, \psi : \{C_n, \partial_n\} \to \{C'_n, \partial'_n\}$ be chain maps. A chain homotopy from ϕ to ψ is a homomorphism $P : C_n \to C'_{n+1}$ such that

$$\phi - \psi = \partial P + P\partial.$$

Key Prop. If ϕ and ψ are chain homotopic then $\phi_* = \psi_* : H_i(\mathcal{C}) \to H_i(\mathcal{C})$ for all i.

Apply this to our situation: letting $\phi = g_\#$, $\psi = h_\#$ gives us what we want.

Proof of Key Prop. Given $c \in Z_i(\mathcal{C})$, we want to prove that $\phi_*([c]) = \psi_*([c])$. This is the case if and only if $\phi(c) - \psi(c) \in B_i(\mathcal{C}')$. We have that $\partial c = 0$, so

$$\phi(c) - \psi(c) = P\partial(c) + \partial P(c) = \partial P(c) \qquad \square.$$

We'd like the map $C_n(X) \hookrightarrow C_n^S(X)$ to induce isomorphisms on homology, but we're missing something deep when we're thinking that.

Developing Singular Homology

The setup of relative (singular) homology is a subspace $A \hookrightarrow X$. This induces an inclusion $C_n(A) \hookrightarrow C_n(X)$. We define $C_n(X, A)$ by

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

(we'll talk about exact sequences next time). This seems like a random algebraic thing to define, but we'll be able to relate $C_n(X, A)$ to $C_n(X/A)$. $\square$

Lecture 7 (2012-10-15)

There will be an in-class midterm the week after next.

Until further notice, $H_i(X)$ will denote the singular homology groups; also could be simplicial, really just pretend I'm doing both at the same time.

We've defined simplicial and singular homology, and proved that singular homology is a homotopy functor. However, we can't compute singular homology. Thus, we want to show that $H_i \cong H_i^S$.

Relative Homology

Suppose that X is a topological space and $A \subset X$ a subspace, with $i : A \hookrightarrow X$ the inclusion. This obviously induces an injective chain map $i_\# : C_n(A) \hookrightarrow C_n(X)$; we'll identify $C_n(A)$ with its image under $i_\#$. There is an induced map on homology $i_* : H_n(A) \to H_n(X)$. Note that i_* is often not injective, for example with $X = \mathbb{D}^2$ and $A = \mathbb{S}^1$, we get $H_1(\mathbb{S}^1) \to H_1(\mathbb{D}^2)$ is a map from $\mathbb{Z}$ to 0.

Definition. The relative chain group is defined to be

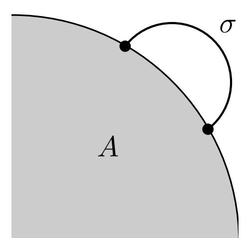
$$C_n(X, A) := C_n(X)/C_n(A).$$

Because ∂_n on $C_n(X)$ preserves $C_n(A)$, we get an induced map $\partial_n' : C_n(X, A) \to C_{n-1}(X, A)$. Then

$$\mathcal{C}'(X, A) = \{C_n(X, A), \partial_n'\}$$

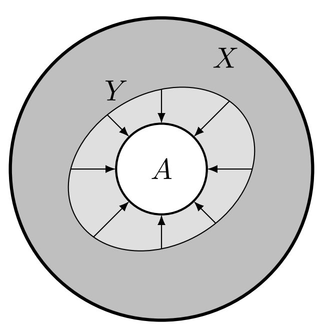
is a chain complex, and we define the relative homology group to be $H_i(X, A) = H_i(\mathcal{C}'(X, A))$.

A cycle $\sigma \in Z_n(X, A)$ is just a chain $\sigma \in C_n(X)$ such that $\partial\sigma \in C_{n-1}(A)$.



Definition. Suppose that $A \subset Y$. Then A is a deformation retract of Y, denoted $Y \searrow A$, if there is a homotopy $F : Y \times [0, 1] \to Y$ such that $F_0 = \mathrm{id}_Y$, $F_1(Y) \subseteq A$, and for all $t \in [0, 1]$, we have $F_t|_A = \mathrm{id}_A$.

Definition. A pair (X, A) of spaces is reasonable if there is a neighborhood $Y \supset A$ such that $Y \searrow A$. This is the case for all Δ-complexes. This rules out crazy things like the topologist's sine curve.



Theorem (Homology of quotient spaces)**.** *Let (X, A) be a reasonable pair. Then the quotient $P : (X, A) \to (X/A, A/A)$ induces an isomorphism for all $i \geq 0$,*

$$P_* : H_i(X, A) \to H_i(X/A, A/A) \cong \widetilde{H}_i(X/A).$$

Fundamental Theorem of Homological Algebra

Let

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \longrightarrow \cdots$$

be a sequence of abelian groups, or chain complexes, or R-modules, ... with ϕ_n appropriate morphisms. The sequence is exact at A_n if $\operatorname{im}(\phi_{n+1}) = \ker(\phi_n)$.

Example. A short exact sequence (SES) is a sequence

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

which is exact at A, B, and C. Being exact at A is equivalent to ϕ being injective, being exact at C is equivalent to ψ being onto. Key example:

$$0 \to \bigoplus C_n(A) \to \bigoplus C_n(X) \to \bigoplus C_n(X, A) \to 0$$

This is a SES of chain complexes.

Theorem (Fundamental Theorem of Homological Algebra)**.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be chain complexes, and suppose that*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a SES of chain complexes. Then there exists a connecting homomorphism $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$ and a "long exact sequence" of abelian groups

$$\cdots \longrightarrow H_n(\mathcal{A}) \xrightarrow{\phi_*} H_n(\mathcal{B}) \xrightarrow{\psi_*} H_n(\mathcal{C}) \xrightarrow{\delta} H_{n-1}(\mathcal{A}) \xrightarrow{\phi_*} H_{n-1}(\mathcal{B}) \longrightarrow \cdots$$

Start of proof. We need to define $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$. The natural candidate is

$$\begin{tikzcd}
Z_n(\mathcal{C}) \arrow[r] \arrow[d] & Z_{n-1}(\mathcal{A}) \arrow[r] & H_{n-1}(\mathcal{A}) \\
H_n(\mathcal{C}) = Z_n(\mathcal{C})/B_n(\mathcal{C}) \arrow[urr, dashed] & &
\end{tikzcd}$$

□

A short exact sequence of chain complexes just means a diagram like this:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\phi_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \arrow[r, "\psi_{n+1}"] \arrow[d, "\partial'_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\phi_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "\psi_n"] \arrow[d, "\partial'_n"] & C_n \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d, "\partial_{n-1}"] & B_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d, "\partial'_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial''_{n-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Lecture 8 (2012-10-17)

Theorem (Fundamental Theorem of Homological Algebra)**.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be chain complexes, and suppose that*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a SES of chain complexes. Then there exists a connecting homomorphism $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$ and a long exact sequence of abelian groups

$$\cdots \longrightarrow H_n(\mathcal{A}) \xrightarrow{\phi_*} H_n(\mathcal{B}) \xrightarrow{\psi_*} H_n(\mathcal{C}) \xrightarrow{\delta} H_{n-1}(\mathcal{A}) \xrightarrow{\phi_*} H_{n-1}(\mathcal{B}) \longrightarrow \cdots$$

Proof. Our SES of chain complexes gives us the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "\phi_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \arrow[r, "\psi_{n+1}"] \arrow[d, "\partial'_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\phi_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "\psi_n"] \arrow[d, "\partial'_n"] & C_n \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\phi_{n-1}"] & B_{n-1} \arrow[r, "\psi_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

We want to define $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$, mapping a homology class $[c]$ to a homology class $[a]$. Note that $H_n(\mathcal{C}) = Z_n(\mathcal{C})/B_n(\mathcal{C})$ and $H_{n-1}(\mathcal{A}) = Z_{n-1}(\mathcal{A})/B_{n-1}(\mathcal{A})$.

There are three key rules to use in any "diagram chase":

1. $\partial^2 = 0$
2. exactness
3. chain maps

Given an n-cycle $c \in C_n$, so that $\partial c = 0$, the fact that ψ is onto implies that there is a $b \in B_n$ such that $\psi(b) = c$. Note that

$$0 = \partial c = \partial\psi(b) = \psi(\partial b) \implies \partial b \in \ker(\psi) = \operatorname{im}(\phi).$$

Thus, there is an $a \in A_{n-1}$ such that $\phi(a) = \partial b$. Check that $a \in Z_{n-1}(\mathcal{A})$:

$$\phi(\partial a) = \partial\phi(a) = \partial\partial b = 0.$$

But ϕ is injective, so $\partial a = 0$.

Thus, given $[c] \in H_n(\mathcal{C})$, we can choose a representative c in $Z_n(\mathcal{C})$ and produce an $a \in Z_{n-1}(\mathcal{A})$. Now we need to check that regardless of the representative we choose, the cycle a represents the same class in homology. In other words, we want to check that the proposed map $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$ is well-defined.

Thus, suppose instead of c we'd chosen $c + \partial\theta$ as a representative of $[c]$, where $\theta \in C_{n+1}$. We want to show that δ will map it to the same $[a]$.

Because ψ is surjective, there is some b' such that $\psi(b+b') = c+\partial\theta$. Thus, in particular $\psi(b') = \partial\theta$, and hence

$$\psi(\partial b') = \partial\psi(b') = \partial\partial\theta = 0.$$

This implies that $\partial b' \in \ker(\psi) = \mathrm{im}(\phi)$, hence $\partial b' = \phi(a')$ for some a'. Our goal is to show that there is an a'' such that $a' = \partial a''$.

Note that $\partial(b+b') = \partial b + \partial b' = \partial b + \partial\partial\theta = \partial b$, hence $\partial(b') = 0$.

To finish, note that

$$\phi((a+a')-a) = \phi(a+a') - \phi(a) = \phi(a') = \partial b' = 0.$$

Now we need to prove the exactness of the long exact sequence (LES). Let's just prove exactness at $H_n(\mathcal{B})$ for example. Given $[b] \in H_n(\mathcal{B})$ such that $\psi_*([b]) = 0$, we want to show that there is some $[a] \in H_n(\mathcal{A})$ such that $\phi_*([a]) = [b]$.

We know that $\partial b = 0$, and because $\psi_*([b]) = 0 \in H_n(\mathcal{C})$, we have that $\psi(b) = \partial c$ for some $c \in C_{n+1}$. Thus $\psi(\partial b) = \partial\psi(b) = \partial\partial c = 0$, so that $\partial b \in \ker(\psi) = \mathrm{im}(\phi)$, and hence there is some a' such that $\phi(a') = \partial b$.

Finish as an exercise. $\square$

Now let's apply this theorem to a pair of spaces (X, A). The SES of chain complexes

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

produces a LES

$$\cdots \to H_{n+1}(X,A) \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

We'll prove later using this tool that if (X, A) is reasonable, then $H_n(X,A) \cong H_n(X/A)$.

Example. Let $(X,A) = (\mathbb{D}^n, \partial\mathbb{D}^n)$ for $n \geq 1$. Note that $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$.

Let $k > 1$. The LES says that

$$H_{k+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \to H_k(\mathbb{S}^{n-1}) \to H_k(\mathbb{D}^n) \to H_k(\mathbb{D}^n, \mathbb{S}^{n-1}) \to H_{k-1}(\mathbb{S}^{n-1}) \to H_{k-1}(\mathbb{D}^n)$$

Any homology of the disk for $k > 1$ is 0, because it is contractible. We can even show that directly in singular homology, because homology is a homotopy functor and the disk is homotopy equivalent to a point, whose singular homology we can compute. Thus, we get for any $k > 1$

$$0 \to H_k(\mathbb{S}^n) \to H_{k-1}(\mathbb{S}^{n-1}) \to 0$$

and this just says that $H_k(\mathbb{S}^n) \cong H_{k-1}(\mathbb{S}^{n-1})$.

Let's look at what happens at the end.

$$\begin{array}{ccccccccc} H_1(\mathbb{D}^n, \mathbb{S}^{n-1}) & \longrightarrow & H_0(\mathbb{S}^{n-1}) & \longrightarrow & H_0(\mathbb{D}^n) & \longrightarrow & H_0(\mathbb{D}^n, \mathbb{S}^{n-1}) & \longrightarrow & 0 \\ \begin{smallmatrix} 0 \text{ if } n>1 \\ \mathbb{Z} \text{ if } n=1 \end{smallmatrix} & & \begin{smallmatrix} \mathbb{Z} \text{ if } n>1 \\ \mathbb{Z}^2 \text{ if } n=1 \end{smallmatrix} & & \mathbb{Z} & & 0 & & \end{array}$$

Our inductive claim is that for all $n \geq 1$ and $k > 1$,

$$H_k(\mathbb{S}^n) = \begin{cases} 0 & \text{if } k \neq 0, n \\ \mathbb{Z} & \text{if } k = 0, n. \end{cases}$$

Here is a table:

	H_0	H_1	H_2
$\mathbb{S}^0$	$\mathbb{Z}^2$	0	0
$\mathbb{S}^1$	$\mathbb{Z}$	$\mathbb{Z}$	0
$\mathbb{S}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$

Lecture 9 (2012-10-19)

Last time we computed that $\widetilde{H}_i(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{if } i \neq n, \end{cases}$, and that $\widetilde{H}_i(\mathbb{D}^n) = 0$ for all i. Thus, as a corollary we can now conclude the Brouwer fixed point theorem.

This theorem is a big deal. Thurston used the Brouwer fixed point theorem to classify homeomorphisms of hyperbolic 3-manifolds up to homotopy by taking the space of hyperbolic metrics, which any homotopy class of homeomorphisms acts on, and looking at the various places a fixed point could be.

Now we'll finally prove that $H_i^{\Delta}(X) \cong H_i^S(X)$.

Let X be a Δ-complex and A a subcomplex of X. There exists a chain map $\psi : C_i^{\Delta}(X,A) \to C_n^S(X,A)$ sending σ to itself. This induces a homomorphism $\psi_* : H_n^{\Delta}(X,A) \to H_n^S(X,A)$.

Theorem. *For all (X,A) and $n \geq 0$, ψ_* is an isomorphism.*

Corollary ("Amazing"). *$H_n^{\Delta}(X,A)$ doesn't depend on which Δ-complex structure you choose!*

Proof. We'll do this for finite-dimensional X and $A = \varnothing$. Read about other cases in Hatcher.

Let X^k = "k-skeleton of X" = $\{$the k-simplices $\sigma_i : \Delta^k \to X\}$.

We have a LES of the pair (X^k, X^{k-1}) in each homology theory, and a commutative diagram comparing them because ψ is a chain map:

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\psi_*"] & H_n^{\Delta}(X^{k-1}) \arrow[r] \arrow[d, "\psi_*"] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d, "\psi_*"] & H_n^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\psi_*"] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[d, "\psi_*"] \\
H_{n+1}^{S}(X^k, X^{k-1}) \arrow[r] & H_n^{S}(X^{k-1}) \arrow[r] & H_n^{S}(X^k) \arrow[r] & H_n^{S}(X^k, X^{k-1}) \arrow[r] & H_{n-1}^{S}(X^{k-1})
\end{tikzcd}$$

We want to induct on k to prove that $H_n^{\Delta}(X^k) \xrightarrow{\cong} H_n^S(X^k)$. If this is true for all k, then we're done, because (by our assumption that X is finite-dimensional), we have $X = X^k$ for some k. The claim is true for $k = 0$ because we know how to look at points. By our inductive hypothesis, the 2nd and 5th vertical arrows in our diagram are isomorphisms.

Now we will prove that the 1st and 4th vertical arrows are isomorphisms. Intuitively, taking the k-skeleton of X (which is just a gluing of k-simplices) and quotienting by the $(k-1)$-skeleton is just crushing the boundaries of all of the k-simplices, turning the k-skeleton into a union and/or wedge of spheres.

We have that

$$C_n^{\Delta}(X^k, X^{k-1}) = \begin{cases} 0 & \text{if } n \neq k, \\ \mathbb{Z}^d & \text{if } n = k. \end{cases}$$

where $d = \#(X^k \setminus X^{k-1})$. Thus, $H_n^{\Delta}(X^k, X^{k-1})$ is the same. Now we want to find $H_i^S(X^k, X^{k-1})$.

Recall that $X^k = \{\sigma_i : \Delta^k \to X^k\}$. Define the map $\Phi : \coprod_i (\Delta^k, \partial\Delta^k) \xrightarrow{\sigma_i} (X^k, X^{k-1})$. Then Φ induces a **homeomorphism**

$$\coprod \Delta^k / \coprod \partial\Delta^k \xrightarrow{\cong} X^k / X^{k-1}$$

essentially by the definition of X being a Δ-complex. This is the same as our proof that

$$(\mathbb{D}^n, \partial\mathbb{D}^n) \to (\mathbb{D}^n/\partial\mathbb{D}^n, \partial\mathbb{D}^n/\partial\mathbb{D}^n) = (\mathbb{D}^n/\partial\mathbb{D}^n, \mathrm{pt})$$

induces an isomorphism in homology. This is the same map that we're using in our big diagram, so **that** map is the isomorphism (a priori, just knowing the domain and codomain are isomorphic doesn't tell us that our map is an isomorphism).

Lemma (Five Lemma)**.** *Given a map between exact sequences*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\eta"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

where $\alpha, \beta, \delta, \eta$ are isomorphisms, then γ is an isomorphism.

Now we're done, as long as we can prove formally

Theorem (Excision)**.** *Given a topological space X and $A, U \subseteq X$ subspace with $\overline{U} \subset \mathrm{int}(A)$, the inclusion $i : (X/U, A/U) \hookrightarrow (X, A)$ induces an isomorphism on homology.*

Proof. We'll do the case when Δ-complex and each of A, $X - U$, $A - U$ is a subcomplex of X.

We get chain maps

$$C_n(X - U) \to C_n(X) \to C_n(X, A)$$

and call the composition ϕ. The map ϕ is surjective, which is clear - any chain in X without any part in A certainly is a chain in X without any part in U. Then ϕ induces an isomorphism.

$$C_n(X - U, A - U) \overset{\text{def}}{=} \frac{C_n(X - U)}{C_n(A - U)} \cong C_n(X, A). \qquad \square$$

Thus, we have proven that Δ-complex homology and singular homology are isomorphic. $\square$

Lecture 10 (2012-10-22)

The midterm will be on November 2nd in class; it will be closed book, and cover everything up to the class before that.

Theorem (Homology of quotients)**.** *Let X be a topological space, and $A \subseteq X$ a reasonable subspace. Then the quotient map $p : (X, A) \to (X/A, A/A)$ induces for all $i \geq 0$ an isomorphism*

$$p_* : H_i(X, A) \to H_i(X/A, A/A) \underset{\text{from homework}}{\cong} \widetilde{H}_i(X/A).$$

This theorem is not as powerful as it may seem, because it only allows us to study smashing the subspace A to a point; we often want to glue two subspaces together without turning them into a single point.

Proof. We will do the case that A is open.

As always, we get the induced map $p_* : H_n(X, A) \to H_n(X/A, A/A)$.

$$\begin{tikzcd} H_n(X, A) \arrow[r, "\cong", "\text{excision}"'] \arrow[d, "p_*"'] & H_n(X - A, A - A) = H - n(X - A) \arrow[d, "(p|_{X-A})_*"] \\ H_n(X/A, A/A) \arrow[r, "\cong"'] & H_n(X/A - A/A) \end{tikzcd}$$

The key is that $(p|_{X-A})$ is a homeomorphism onto its image. □

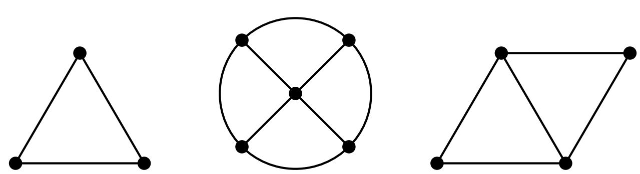
We want to be able compute the homology of $\mathbb{R}P^n$, $\mathbb{C}P^n$, a knot $\mathbb{S}^3 - K$, and other interesting spaces.

Two applications of what we've done so far include the Euler characteristic and the Lefschetz Fixed Point Formula.

Definition. Let X be a finite Δ-complex. Let $c_n(X)$ denote the rank of $C_n^\Delta(X)$. The Euler characteristic of X is

$$\chi(X) = \sum_{n \geq 0} (-1)^n c_n.$$

Let's look at some Δ-complex structures on the disk:



0-simplices	1-simplices	2-simplices	$\#0 - \#1 + \#2$
3	3	1	1
5	8	4	1
4	5	2	1

Theorem (Topological invariance of χ)**.** *Let X be a finite Δ-complex. Then*

$$\chi(X) = \sum_{n \geq 0} (-1)^n b_n(X)$$

where $b_n(X)$ is the nth Betti number of X,

$$b_n(X) = \operatorname{rank}(H_n(X)/T_n(X)).$$

In particular, because $X \simeq Y \implies b_n(X) = b_n(Y)$ for all $n \geq 0$, the right side, and hence the left side, depends only on the homotopy type of X.

The essential idea is that (letting lowercase letters denote the ranks of the groups typically denoted by the corresponding uppercase letters):

$$h_n = z_n - b_n, \quad c_n = z_n + b_{n-1}$$

and using this in the alternating sum makes the proof pop out.

Hopf Trace Formula

Let $\mathcal{C} = \{C_n, \partial_n\}$ be a finite chain complex of finitely generated free abelian groups. Let $f : \mathcal{C} \to \mathcal{C}$ be a chain map. We know that $C_n \cong \mathbb{Z}^d$ for some d, so picking a basis for C_n, we can think of each $f_n \in \mathrm{M}_d(\mathbb{Z})$. Then $\operatorname{tr}(f_n)$ is the trace of this matrix. Recall that

$$\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(B),$$

so $\operatorname{tr}(f_n)$ doesn't depend on choice of basis.

We also have that $(f_n)_*$ acts on $H_n(\mathcal{C})/T_n(\mathcal{C})$, where T_n denotes the torsion subgroup of $H_n(\mathcal{C})$. Thus, we can also consider $\operatorname{tr}((f_n)_*)$.

Theorem (Hopf Trace Formula)**.** *We have*

$$\sum_{n\geq 0}(-1)^n \operatorname{tr}(f_n) = \sum_{n \geq 0}(-1)^n \operatorname{tr}((f_n)_*)$$

Remark. Let $C_n = C_n^\Delta(X)$, and $f = (\mathrm{id}_X)_\#$. Then we get that

$$\sum_{n\geq 0}(-1)^n \operatorname{rank}(C_n^\Delta(X)) = \sum_{n\geq 0}(-1)^n b_n(X).$$

Proof. We have $B_n \subseteq Z_n \subseteq C_n$, all free abelian groups. First, pick a basis $\partial\sigma_1, \ldots, \partial\sigma_r$ for B_n; extend to (**not a basis; a maximal $\mathbb{Z}$-linearly independent set**) $z_1, \ldots, z_s \in Z_n$; then extend to a basis of C_n. Thus, we have a basis for C_n that we've broken up into boundaries, plain old chains, and cycles.

We want to compute, for each element of the basis above, the coefficient of v in $f_n(v)$. Let's call this $\lambda(f_n(v))$.

Because f_n is a chain map,

$$\lambda(f_n(\partial_{n+1}\sigma_j)) = \lambda(f_{n+1}(\sigma_j))$$

Then

$$\sum_{n\geq 0}(-1)^n \operatorname{tr}(f_n) = \sum_{n\geq 0}(-1)^n \sum_{k=0}^{n} \lambda(f(z_k))$$

because all other terms cancel in pairs. Because the action of f_* on $[z_i] \in H_i$ is just sending it to $[f(z_i)]$, the coefficents of the action of f_* and of f itself are the same. Thus, the left side is equal to

$$\sum_{n\geq 0}(-1)^n \operatorname{tr}((f_n)_*).$$

□

Lecture 11 (2012-10-24)

Neat fact: given closed $A_1, \dots, A_{n+1} \subset \mathbb{S}^n$ such that $\bigcup A_i = \mathbb{S}^n$, there exists a j such that A_j contains a pair of antipodal points.

The Lefschetz Fixed Point Theorem

Last time, we talked about the Hopf trace formula, where we had a complex $\mathcal{C} = \{C_n, \partial_n\}$ and a chain map $\varphi : \mathcal{C} \to \mathcal{C}$, which induces maps $(\varphi_n)_* : H_n(\mathcal{C})/T_n(\mathcal{C}) \circlearrowleft$, and deduced that

$$\sum (-1)^n \operatorname{tr}(\varphi_n) = \sum (-1)^n \operatorname{tr}((\varphi_n)_*).$$

Let's do some setup first. Let X be a finite Δ-complex, and let $f : X \to X$ be a continuous map. Consider $(f_n)_* : H_n(X)/T_n(X) \circlearrowleft$.

Definition. The Lefschetz number of f is

$$\Lambda(f) = \sum_{n \geq 0} (-1)^n \operatorname{tr}((f_n)_*).$$

Theorem (LFPT). *If $\Lambda(f) \neq 0$, then f has a fixed point.*

Corollary. *Let X be a contractible Δ-complex, e.g. $X = \mathbb{D}^n$. Then any $f : X \to X$ has a fixed point.*

Proof of Corollary. By hypothesis, we have that X is "aspherical", i.e.

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n > 0 \end{cases}$$

This implies that $(f_n)_* = 0$ for all $n > 0$, and hence $\operatorname{tr}((f_n)_*) = 0$ for all $n > 0$. We also have (from your current homework)

$$\operatorname{tr}((f_0)_*) = \operatorname{tr}(\mathrm{id}) = \operatorname{tr}([1]) = 1.$$

Thus,

$$\Lambda(f) = 1 - 0 + 0 - \cdots = 1 \neq 0.$$ □

Proof of Lefschetz. Suppose that $f(x) \neq x$ for all $x \in X$.

Put a metric d on X. There is some $\delta > 0$ such that $d(f(x), x) > \delta$ for all $x \in X$ because f is continuous, $f(x) \neq x$ for all x, and because X is compact. Now, by barycentrically subdividing, we can put a Δ-complex structure on X such that $\operatorname{diam}(\sigma) < \frac{d}{100}$ for all simplices σ.

By simplicial approximation (which we haven't covered in detail, but you should know the statement at least), we have that, possibly after further subdividing, there is a simplicial map $h : X \to X$ such that $h \sim f$ and $d(h(x), f(x)) \leq \frac{\delta}{2}$ for all $x \in X$.

Thus, for all $x \in X$, we have $d(h(x), x) > \frac{\delta}{10} > \frac{\delta}{100}$, and because h is simplicial, we have $h(\sigma) \cap \sigma = \varnothing$ for all simplices σ.

We claim that this implies $\mathrm{tr}((h_n)_\#) = 0$ for all $n \geq 0$. The matrix for $(h_n)_\#$ acting on $C_n(X) \cong \mathbb{Z}^d$ looks like

$$\begin{array}{c c} & \begin{array}{cccc} \sigma_1 & \sigma_2 & \cdots & \sigma_d \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_d \end{array} & \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \end{array}$$

so, applying the Hopf trace formula,

$$\Lambda(h) = \sum (-1)^n \mathrm{tr}((h_n)_*) = \sum (-1)^n \mathrm{tr}((h_n)_\#) = 0.$$

But $f \sim h$, so $f_* = h_*$, and so $\Lambda(f) = \Lambda(h) = 0$. □

Corollary (of Brouwer)**.** *Let A be an $n \times n$ matrix with real entries $a_{ij} > 0$. Then there exists a real eigenvalue λ with a real eigenvector $(v_1, \dots, v_n)$ with all $v_i > 0$.*

Proof. This has applications to the adjacency matrix of a graph, and to probability.

Let $X = \{\text{rays through } 0 \text{ in a positive orthant}\}$, which is homeomorphic to Δ^{n-1}. We have that A maps X to X by hypothesis. Then Brouwer implies that there exists an $x \in X$ such that $Ax = x$, i.e. there is a ray $\mathbb{R}\vec{x}$ such that $A\vec{v} = \lambda\vec{v}$ Then we must have $\lambda > 0$ and $v_i > 0$ for all coordinates of $\vec{v} = (v_1, \dots, v_n)$. □

Remark (Weil's Observation)**.** Suppose that $f : X \to X$ is a homeomorphism of compact manifolds.

Assumption: Suppose that $\Lambda(f) = \#$ of fixed points of f, and that $\Lambda(f^n) = \#$ of periodic points of f of period n, i.e. points with $f^n(x) = x$.

Claim: There exist algebraic integers $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$ such that

$$\Lambda(f^n) = \# \text{ of periodic points of } f \text{ of period } n = \sum \alpha_i^n - \sum \beta_j^n.$$

The amazing thing is that the left side is obviously an integer, but the right side rarely is if you choose arbitrary α_i and β_i.

The proof is just that

$$\Lambda(f^n) = \sum_i (-1)^i \mathrm{tr}((f_i^n)_*)$$

where $(f_i^n)_* : H_n(X)/T_n(X) \circlearrowleft$, and $H_n(X)/T_n(X) \cong \mathbb{Z}^d$, so that $(f_i)_* \in \mathrm{M}_{d\times d}(\mathbb{Z})$ and hence its eigenvalues $\lambda_1, \dots, \lambda_s$ are algebraic integers, so that $\mathrm{tr}((f_i^n)_*) = \sum \lambda_i^n$.

Moreover, he observed that, if X is a projective smooth variety defined over $\mathbb{Z}$, then

$$X(\mathbb{F}_{p^n}) = \sum \alpha_i^n - \sum \beta_j^n$$

and he conjectured there was a relationship. Indeed, $X(\overline{\mathbb{F}_p}) \circlearrowleft \mathrm{Frob}_p$, and $\mathrm{fix}(\mathrm{Frob}_p^n) = X(\mathbb{F}_{p^n})$.

$$\sum (-1)^i \, \mathrm{Frob}_p^n \circlearrowright \underbrace{H^i_{\mathrm{etal}}(X)}_{\cong H^i(X(\mathbb{C}))}$$

Lecture 12 (2012-10-26)

As a reminder, the midterm is Friday, November 2, 11am - 12:30pm.

Maps of Spheres

A map $f : \mathbb{S}^n \to \mathbb{S}^n$ induces a map $f_* : H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$. Recall that $H_n(\mathbb{S}^n) \cong \mathbb{Z}$, having $[\Delta_1 - \Delta_2]$ as a generator, where Δ_1 and Δ_2 are n-simplices glued along their boundaries to form the "equator" of the sphere.

Definition. The degree of f is the unique $\deg(f) \in \mathbb{Z}$ such that $f_*(z) = \deg(f)z$ for all $z \in H_n(\mathbb{S}^n)$.

Note that

$$\deg(f \circ g)z = (f \circ g)_*(z) = f_*(g_*(z)) = f_*(\deg(g)z) = \deg(g)f_*(z) = \deg(f)\deg(g)z,$$

and hence $\deg(f \circ g) = \deg(f)\deg(g)$. Also, if $f \sim g$, then $f_* = g_*$, and hence $\deg(f) = \deg(g)$. In fact, the converse is true; that is, considering the map

$$\{f : \mathbb{S}^n \to \mathbb{S}^n\}/\sim \;\longrightarrow\; \mathbb{Z}$$

sending $[f]$ to $\deg(f)$, this map is a bijection. This is an old theorem:

Theorem (Hopf). *For any $f, g : \mathbb{S}^n \to \mathbb{S}^n$, then $f \sim g \iff \deg(f) = \deg(g)$.*

Here are some basic properties.

- If f is not surjective, then $\deg(f) = 0$. This is because there is an x such that $f(\mathbb{S}^n) \subseteq \mathbb{S}^n - x$, and $\mathbb{S}^n - x \cong \mathbb{R}^n$, so the fact that we have a commutative diagram

$$\begin{tikzcd} H_n(\mathbb{S}^n) \arrow[r, "f_*"] \arrow[d, "f_*"'] & H_n(\mathbb{S}^n) \\ H_n(\mathbb{S}^n - x) = H_n(\mathbb{R}^n) = 0 \arrow[ur] & \end{tikzcd}$$

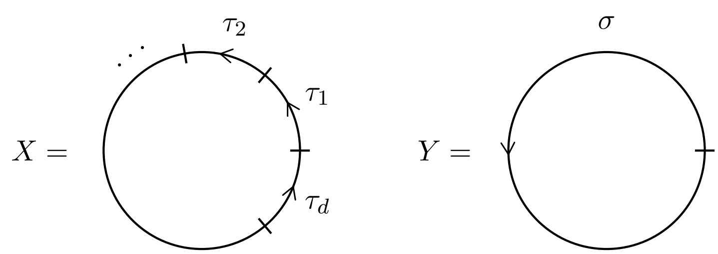
  forces $f_* = 0$.

- We can consider the suspension of a map between spheres, $\Sigma f : \Sigma\mathbb{S}^n \to \Sigma\mathbb{S}^n$. Note that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$ (this is clear if you draw a picture). We get a commutative diagram

$$\begin{tikzcd} H_n(\mathbb{S}^n) \arrow[r, "\cong", "i"'] \arrow[d, "f_*"'] & H_{n+1}(\mathbb{S}^{n+1}) \arrow[d, "\Sigma f_*"] \\ H_n(\mathbb{S}^n) \arrow[r] & H_{n+1}(\mathbb{S}^{n+1}) \end{tikzcd}$$

  implying that $\deg(f) = \deg(\Sigma f)$.

- Let $d \geq 1$. Let $\psi_d : \mathbb{S}^1 \to \mathbb{S}^1$ be the map $z \mapsto z^d$. We give $\mathbb{S}^1$ two different Δ-complex structures,



Then $\psi_d(\tau_i) = \sigma$ for all i, so that the map

$$(\psi_d)_* : \underbrace{H_1(X)}_{\left\langle \left[\sum_{i=1}^d \tau_i \right] \right\rangle} \to \underbrace{H_1(Y)}_{\langle [\sigma] \rangle}$$

satisfies

$$\begin{aligned} (\psi_d)_*([\textstyle\sum_{i=1}^d \tau_i]) &= [(\psi_d)_\# \textstyle\sum_{i=1}^d \tau_i] \\ &= [\textstyle\sum_{i=1}^d \psi_d(\tau_i)] \\ &= [\textstyle\sum_{i=1}^d \sigma] \\ &= d[\sigma] \end{aligned}$$

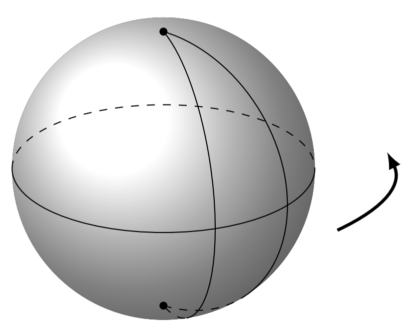
and hence $\deg(\psi_d) = d$. Note that we implicitly used that this diagram commutes:

$$\begin{tikzcd} H_1^\Delta(X) \arrow[r, "f_*^\Delta"] \arrow[d, "\phi"', "\cong"] & H_1^\Delta(Y) \arrow[d, "\cong"', "\phi"] \\ H_1^S(\mathbb{S}^1) \arrow[r, "f_*^S"'] & H_1^S(\mathbb{S}^1) \end{tikzcd}$$

but this is just because ϕ (the isomorphism between simplicial and singular homology) is natural, i.e. the composition of degrees doesn't depend on the chosen Δ-complex structure.

Thus, for all $n \geq 1$ and $d \in \mathbb{Z}$ there is a map $f : \mathbb{S}^n \to \mathbb{S}^n$ of degree d, namely $\Sigma^{n-1}\psi_d$.

In differential topology, while you can run into trouble with a few bad points, you can generically get the degree in a similar way. For example, consider the suspension $\Sigma\psi_d$, which is the map from $\mathbb{S}^2$ to $\mathbb{S}^2$ wrapping the sphere around itself so that a lune of angle $\frac{2\pi}{d}$ is wrapped over the entire sphere.



Then the preimage of almost every point is a set of d points, except the north and south poles, which have only one preimage (they are preserved).

- Let $\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = 1\} \subset \mathbb{R}^{n+1}$. Let $r : \mathbb{S}^n \to \mathbb{S}^n$ be the reflection map

$$r(x_1, x_2 \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1}).$$

We claim that $\deg(r) = -1$. Here is the proof: consider the Δ-complex $X = \Delta_1^n \sqcup \Delta_2^n$ where $\partial\Delta_1^n = \partial\Delta_2^n$. Note that $X \cong \mathbb{S}^n$. Then the induced map $r_\# : C_n(X) \to C_n(X)$ just swaps the simplices, $\Delta_1 \mapsto \Delta_2$ and $\Delta_2 \mapsto \Delta_1$. Also, note that $H_n(X) \cong \mathbb{Z} = \langle [\Delta_1 - \Delta_2] \rangle$. Thus,

$$r_*([\Delta_1 - \Delta_2]) = [r_\#(\Delta_1 - \Delta_2)] = [\Delta_2 - \Delta_1] = -[\Delta_1 - \Delta_2]$$

and hence $\deg(r) = -1$.

As a corollary, this implies that the antipodal map $A(x_1, \dots, x_{n+1}) = (-x_1, \dots, -x_{n+1})$ has degree $(-1)^{n+1}$, because

$$\deg(A) = \deg(r_1 \circ r_2 \circ \cdots \circ r_{n+1}) = \deg(r_1)\deg(r_2)\cdots\deg(r_{n+1}) = (-1)^{n+1},$$

and hence $A \not\sim \mathrm{id}_{\mathbb{S}^n}$ when n is even, and $A \sim \mathrm{id}_{\mathbb{S}^n}$ when n is odd.

Which maps $f : \mathbb{S}^n \to \mathbb{S}^n$ have no fixed points? Well, the antipodal map A, obviously; any others?

The Lefschetz number of any map $f : \mathbb{S}^n \to \mathbb{S}^n$ is

$$\Lambda(f) = \underbrace{\mathrm{tr}(f_* : H_0(\mathbb{S}^n) \circlearrowleft)}_{=1} + (-1)^n \deg(f)$$

Note that if f has no fixed points, then we must have $\Lambda(f) = 0$, hence $\deg(f) = (-1)^{n+1}$, and therefore $f \sim A$ by Hopf's theorem. Thus, $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points implies that $f \sim A$.

Definition. Consider $\mathbb{S}^{n-1}$ as a subset of $\mathbb{R}^n$. A continuous vector field on $\mathbb{S}^{n-1}$ is a continuous map $v : \mathbb{S}^{n-1} \to \mathbb{R}^n$ such that $v(z) \perp z$ for all $z \in \mathbb{S}^{n-1}$.

Theorem. *The n-sphere $\mathbb{S}^n$, for $n \geq 1$, admits a nowhere zero vector field $v \iff n$ is odd.*

Proof. To see $\Longleftarrow$, consider $v(x_1, \dots, x_{n+1}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{n+1}, x_n)$. We needed n odd to be able to pair off the coordinates.

To see $\Longrightarrow$, use the vector field v to prove $\mathrm{id}_{\mathbb{S}^{n-1}} \sim A$. If v is our nowhere vanishing vector field, then consider the map $v_t : \mathbb{S}^{n-1} \times I \to \mathbb{S}^{n-1}$ defined by

$$v_t(z) = -\cos(\pi t)z + \sin(\pi t)v(z).$$

As t goes from 0 to 1, this map slides a point $z \in \mathbb{S}^n$ along the great circle connecting z to $-z$ in the direction determined by $v(z)$. This is a homotopy from $\mathrm{id}_{\mathbb{S}^{n-1}}$ to A, so $\deg(\mathrm{id}_{\mathbb{S}^{n-1}}) = 1 = (-1)^{n+1} = \deg(A)$, hence n is odd. $\square$

Theorem (Adams, 1962)**.** *Let $n \geq 1$, and write $n + 1 = 2^{4a+b}(2k + 1)$ where $a, b, k \in \mathbb{Z}$, and $0 \leq b \leq 3$. Then the maximum number of linearly independent vector fields on $\mathbb{S}^n$ is precisely $2^b + 8a - 1$.*

Lecture 13 (2012-10-29)

CW-Complexes and Cellular Homology

Definition. CW-complexes are built up in an inductive process. The base of the inductive process is $X^{(0)}$, the 0-skeleton, consisting of discrete points. The inductive step is as follows: assuming we have constructed $X^{(n-1)}$, we are given

- a collection of n-balls $\{\mathbb{D}^n_\alpha \mid \alpha \in I\}$,
- maps $\{\phi_\alpha : \partial\mathbb{D}^n_\alpha \to X^{(n-1)}\}$ attaching the boundaries of the balls to the $(n-1)$-skeleton.

Then we define

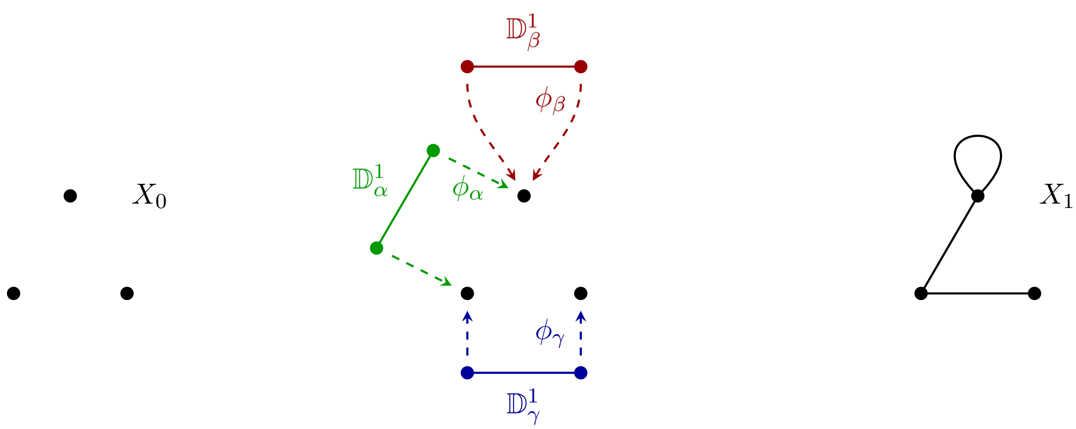
$$X^{(n)} = \frac{X^{(n-1)} \sqcup \coprod_{\alpha\in I} \mathbb{D}^n_\alpha}{x \sim \phi_\alpha(x) \text{ for all } \alpha \in I, x \in \partial\mathbb{D}^n_\alpha}.$$

The space $X = \bigcup_{n\geq 0} X^{(n)}$ built in this manner is called a CW-complex. When $X \cong X^{(N)}$, we say that X is N-dimensional.

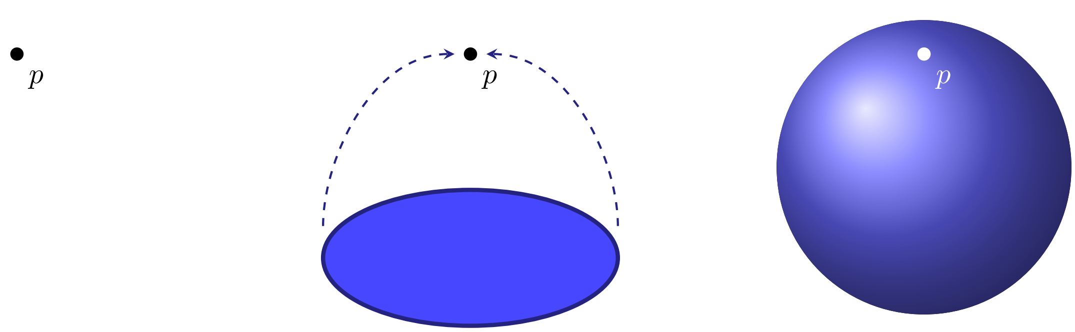
Note that we give X the weak topology, i.e. we declare $A \subseteq X$ to be open precisely when $\phi_\alpha^{-1}(A) \subseteq \mathbb{D}^n_\alpha$ is open for all cells $\phi_\alpha : \mathbb{D}^n_\alpha \to X$.

Examples.

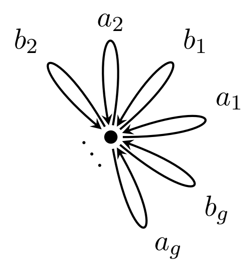
- A 1-dimensional CW-complex is equivalent to a graph. Here is an example:



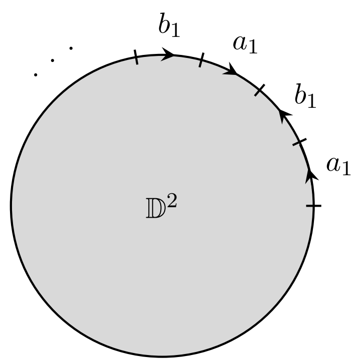
- The n-sphere $\mathbb{S}^n$ can be given a CW-complex structure with one 0-cell p and one n-cell ϕ, by letting $\phi : \mathbb{D}^n \to \{p\}$ be the constant map.



- The surface Σ_g of genus $g \geq 1$ can be given a CW-complex structure with one 0-cell v, $2g$ 1-cells $a_1, b_1, \ldots, a_g, b_g$, and one 2-cell $\phi : \partial\mathbb{D}^2 \to X^{(1)}$ defined by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. The 1-skeleton just looks like



We attach $\partial\mathbb{D}^2$ to the 1-skeleton $X^{(1)}$ as follows. We break up $\partial\mathbb{D}^2$ into $4g$ arcs of angle $\frac{2\pi}{4g}$,

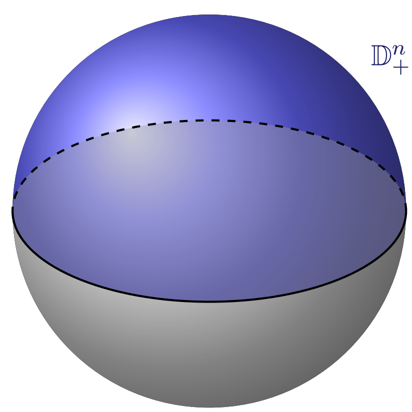


and define ϕ by mapping an arc to the indicated edge in the 1-skeleton, with degree ± 1 depending on the sign determined by the word (this is reflected in the orientations of the arcs in the picture).

- Real projective space $\mathbb{R}P^n$ is defined to be

$$\{\text{lines through origin in } \mathbb{R}^{n+1}\} = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\substack{u \sim v \text{ iff } u = rv \\ \text{for some } r \in \mathbb{R}^\times}} = \frac{\{v \in \mathbb{R}^{n+1} \setminus \{0\} : \|v\| = 1\}}{v \sim -v} = \frac{\mathbb{S}^n}{v \sim -v}.$$

Denoting the upper hemisphere of $\mathbb{S}^n$ by $\mathbb{D}^n_+$,



then we can also see that

$$\mathbb{R}\mathrm{P}^n = \frac{\mathbb{D}^n_+}{v \sim -v \text{ for all } v \in \partial\mathbb{D}^n_+}.$$

Note that $\partial\mathbb{D}^n_+ = \mathbb{S}^{n-1}$. We have the quotient map $\phi : \mathbb{S}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ identifying v and $-v$ for all $v \in \mathbb{S}^{n-1}$; this is just the attaching map for the n-cell $\mathbb{D}^n_+$ when we give $\mathbb{R}\mathrm{P}^n$ a CW-complex structure.

In general, $\mathbb{R}\mathrm{P}^n$ is a CW-complex with 1 i-cell for $i = 0, 1, \ldots, n$. Thus, $\mathbb{R}\mathrm{P}^0$ is just a point, $\mathbb{R}\mathrm{P}^1$ is $\mathbb{S}^1$, and

$$\mathbb{R}\mathrm{P}^n = \frac{\mathbb{R}\mathrm{P}^{n-1} \sqcup \mathbb{D}^n_+}{v \sim \phi(v) \text{ for all } v \in \partial\mathbb{D}^n}.$$

How to Compute CW Homology

Let X be a CW-complex. We define a chain complex $\mathcal{C}^{\mathrm{CW}}(X) = \{C^{\mathrm{CW}}_n(X), d_n\}$ by letting $C^{\mathrm{CW}}_n(X)$ be the free abelian group on the n-cells, and the boundary maps $d_n : C^{\mathrm{CW}}_n(X) \to C^{\mathrm{CW}}_{n-1}(X)$ satisfy $d_{n-1} \circ d_n = 0$, but we'll skip their definition to get to some calculations.

Of course, lastly we define $H^{\mathrm{CW}}_i(X) = H_i(\mathcal{C}^{\mathrm{CW}}(X))$.

Examples.

- Let's compute the CW homology of $\mathbb{S}^n$, for $n \geq 2$. The CW chain complex is just

$$\begin{array}{ccccccccccc} C^{\mathrm{CW}}_{n+1}(\mathbb{S}^n) & \longrightarrow & C^{\mathrm{CW}}_n(\mathbb{S}^n) & \longrightarrow & C^{\mathrm{CW}}_{n-1}(\mathbb{S}^n) & \longrightarrow & \cdots & \longrightarrow & C^{\mathrm{CW}}_0(\mathbb{S}^n) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

  and therefore

$$H^{\mathrm{CW}}_i(\mathbb{S}^n) \cong C^{\mathrm{CW}}_i(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

  This agrees with singular, and hence also simplicial, homology.

- The CW chain complex for Σ_g is

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z}^{2g} & \xrightarrow{d_1} & \mathbb{Z} & \longrightarrow & 0 \\ & & & & \parallel & & \parallel & & \parallel & & \\ & & & & \langle f \rangle & & \langle a_1, b_1, \ldots, a_g, b_g \rangle & & \langle v \rangle & & \end{array}$$

  Every a_i and b_i is sent to $v - v = 0$, so that $d_1 = 0$. Now note that d_2 sends f to

$$a_1 + b_1 - a_1 - b_1 + a_2 + \cdots + a_g + b_g - a_g - b_g = 0$$

  and therefore $d_2 = 0$. Thus,

$$H^{\mathrm{CW}}_i(\mathbb{S}^n) \cong C^{\mathrm{CW}}_i(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } 2, \\ \mathbb{Z}^{2g} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

  Morally, the boundary map in CW homology is measuring degree of the attaching maps of the cells.

Lecture 14 (2012-10-31)

The midterm will be Friday, November 2 from 11am to 12:30pm, in Ryerson 352 (the Barn).

Cellular Homology

Proposition. *Let X be a CW-complex. Then for all $n \geq 0$,*

1. $\widetilde{H}_i(X^{(n)}, X^{(n-1)}) = \begin{Bmatrix} \mathbb{Z}^d & \text{if } i = n \\ 0 & \text{if } i \neq n \end{Bmatrix}$ *where d is the number of n-cells in X.*
2. $\widetilde{H}_i(X^{(n)}) = 0$ *for all $i > n$.*
3. *The inclusion $X^{(n)} \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^{(n)}) \to H_k(X)$ for all $k < n$.*

Proof. Consider the LES of the pair $(X^{(n)}, X^{(n-1)})$. Also, observe that $\widetilde{H}_i(X^{(n)}, X^{(n-1)}) \cong \widetilde{H}_i(X^{(n)}/X^{(n-1)})$, and $X^{(n)}/X^{(n-1)}$ is a wedge of spheres. Lastly, we have a commutative diagram

$$\begin{tikzcd} \mathbb{D}^n_\alpha \arrow[r] \arrow[d] \arrow[dr] & X^{(n)} = \dfrac{X^{(n-1)} \sqcup \coprod_\alpha \mathbb{D}^n_\alpha}{x \sim \phi_\alpha(x) \text{ for all } x \in \partial\mathbb{D}^n_\alpha} \arrow[d] \\ \mathbb{D}^n_\alpha/\partial\mathbb{D}^n_\alpha \arrow[r] & X^{(n)}/X^{(n-1)} \cong \bigvee \mathbb{S}^n \end{tikzcd}$$

□

Definition. The group of cellular n-chains on X is defined to be $C_n^{\text{CW}}(X) := H_n(X^{(n)}, X^{(n-1)})$. Thus, $C_n^{\text{CW}}(X) \cong \mathbb{Z}^d$ where d is the number of n-cells in X.

Our goal is to define the boundary homomorphisms $d_n : C_n^{\text{CW}}(X) \to C_{n-1}^{\text{CW}}(X)$ and show that they satisfy $d_{n-1} \circ d_n = 0$.

Thinking about the definition of C_n^{CW}, we recall that in the LES of $(X^{(n)}, X^{(n-1)})$, we had a boundary map $\partial_n : H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)})$.

We can also consider the map $(i_{n-1})_* : H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n)}, X^{(n-1)})$.

Now, we define $d_n = i_{n-1} \circ \partial_n$, which has the right domain and range: d_n goes from $H_n(X^n, X^{(n-1)}) = C_n^{\text{CW}}(X)$ to $H_{n-1}(X^{(n-1)}, X^{(n-2)}) = C_{n-1}^{\text{CW}}(X)$.

You should check on your own $d_{n-1} \circ d_n = 0$ for all $n \geq 0$. Thus, $\mathcal{C}^{\text{CW}}(X) = \{C_n^{\text{CW}}(X), d_n\}$ is a chain complex.

Definition. We define the CW homology of X to be $H_n^{\text{CW}}(X) := H_n(\mathcal{C}^{\text{CW}}(X))$.

Theorem. *Let X be any CW-complex. Then $H_n^{CW}(X) \cong H_n(X)$ for all $n \geq 0$.*

Corollary. *We then have that*

- $H_n(X) = 0$ *for all $n > \dim(X)$.*
- $\operatorname{rank}(H_n(X)) \leq \#$ *of n-cells of X.*
- $H_n^{CW}(X)$ *is a homotopy invariant.*

Remark. By the topological invariance of H_n^{CW} and the Hopf trace formula, we can compute

$$\chi(X) = \sum_{n\geq 0} (-1)^n \operatorname{rank}(C_n^{\mathrm{CW}}(X)).$$

For example,

$$\chi(\mathbb{S}^n) = 1 + (-1)^n = \begin{cases} 0 & \text{if } n \text{ odd}, \\ 2 & \text{if } n \text{ even}. \end{cases}$$

$$\chi(\mathbb{R}P^n) = 1 - 1 + 1 - \cdots + (-1)^n = \begin{cases} 0 & \text{if } n \text{ odd}, \\ 1 & \text{if } n \text{ even}. \end{cases}$$

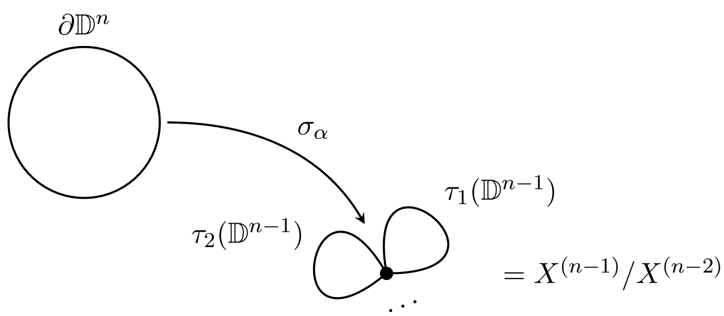
$$\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g.$$

To compute $H_n^{\mathrm{CW}}(X)$, we need to understand the boundary maps d_n. We have

$$C_n^{\mathrm{CW}}(X) = \mathbb{Z}\text{-span of } \{\sigma_\alpha : \mathbb{D}_\alpha^n \to X^{(n)}\}$$

$$C_{n-1}^{\mathrm{CW}}(X) = \mathbb{Z}\text{-span of } \{\tau_\beta : \mathbb{D}_\beta^{n-1} \to X^{(n-1)}\}$$

Consider the composition of the attaching map of an n-cell σ_α with the quotient map sending $X^{(n-1)}$ to $X^{(n-1)}/X^{(n-2)}$:



Then, for all β, consider the composition of that with the quotient map from $X^{(n-1)}/X^{(n-2)}$ to just one of the $(n-1)$-cells τ_β:

$$\mathbb{S}^{n-1} = \partial\mathbb{D}_\alpha^n \xrightarrow{\sigma_\alpha|_{\partial\mathbb{D}_\alpha^n}} X^{(n-1)} \xrightarrow{\text{quotient}} X^{(n-1)}/X^{(n-2)} \xrightarrow{\substack{\text{more}\\ \text{quotient}}} \tau_\beta(\mathbb{D}^{n-1}) = \mathbb{S}^{n-1}$$

This is a map $\psi_{\alpha,\beta} = \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$.

Lemma (Hatcher, p.141)**.** *For all α, β, we have*

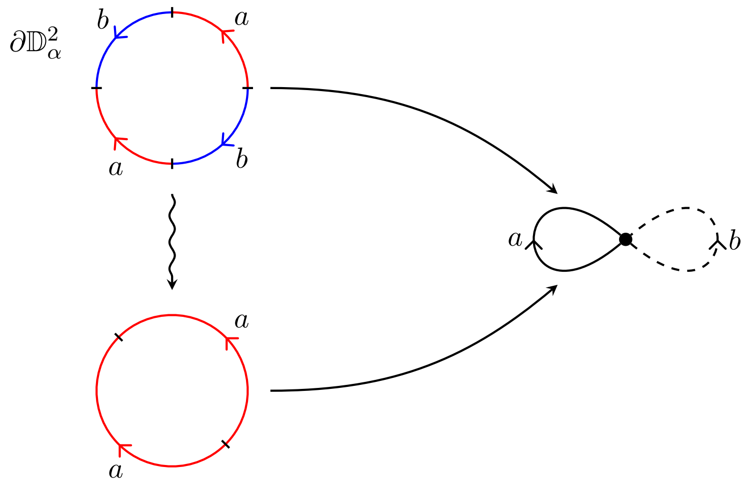
$$d_n(\sigma_\alpha) = \sum_\beta d_{\alpha,\beta}\tau_\beta$$

where $d_{\alpha,\beta} = \deg(\psi_{\alpha,\beta})$.

Let's work out the homology of the torus this way. Let $X = \mathbb{T}^2$. The CW chain complex is

$$\begin{array}{ccccccc} C_2 & & C_1 & & C_0 & & \\ \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z}^2 & \xrightarrow{d_1} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

We obviously have $d_1 = 0$. Now let's think about $\psi_{\alpha a}$. It may be helpful to note that the map sending $\partial\mathbb{D}^2_\alpha$ to $\tau_a(\mathbb{D}^1)$ factors through the quotient collapsing the b arcs:



We clearly have that $\deg(\psi_{\alpha a}) = 1 - 1 = 0$, and similarly, $\deg(\psi_{\alpha b}) = 0$. Therefore $d_2 = 0$, and

$$H_i^{\text{CW}}(\mathbb{T}^2) \cong C_i^{\text{CW}}(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ \mathbb{Z}^2 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Lecture 15 (2012-11-02)

Midterm in Ryerson 352 (the Barn) from 11am to 12:30pm.

Lecture 16 (2012-11-05)

Today we'll discuss Mayer-Vietoris, which is one of the most powerful computational tools we have, and we'll be moving to cohomology soon.

Let X be a space and consider injective maps $i : A \hookrightarrow X$ and $j : B \hookrightarrow X$ (we may as well consider A and B as subsets of X) with the property that $A \cup B = X$, and (X, A) and (X, B) are good pairs.

Theorem (Mayer-Vietoris)**.** *There is a long exact sequence*

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

Proof. There is a short exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(A \cap B) & \xrightarrow{i_\# \oplus j_\#} & C_n(A) \oplus C_n(B) & \xrightarrow{\phi} & C_n(A+B) & \longrightarrow & 0 \\ & & z & \longmapsto & (i(z), j(z)) & & & & \\ & & & & (x, y) & \longmapsto & x - y & & \end{array}$$

where $C_n(A+B)$ is the subgroup of $C_n(X)$ consisting of chains of the form $\sum a_i \sigma_i + \sum c_i \tau_i$ where $\sigma_i \in C_n(A)$ and $\tau_i \in C_n(B)$.

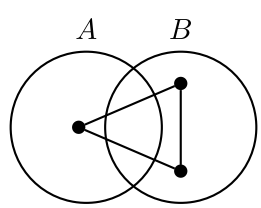
The fundamental theorem of homological algebra implies that we get a long exact sequence

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(A+B) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$$

where $H_n(A+B)$ is the nth homology of the chain complex $\{C_n(A+B)\}$.

The key claim is that the inclusion $h : C_n(A+B) \to C_n(X)$ induces an isomorphism on homology, $h_* : H_n(A+B) \xrightarrow{\cong} H_n(X)$ for all $n \geq 0$. The idea of the proof of this key claim is as follows.

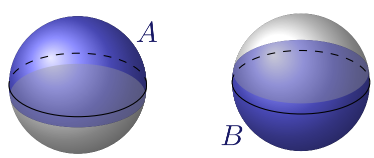
If X is a Δ-complex and we can triangulate X such that the triangulation restricts to triangulations on A, B, and $A \cap B$, then the key claim is easy since $C_n^\Delta(A+B) \xrightarrow{\cong} C_n^\Delta(X)$. In general, we need to subdivide X (use simplicial approximation), because for example, we'd run into trouble with a simplex like this:



$\square$

Applications.

- Let $X = \mathbb{S}^n$, $A = \mathbb{D}^n_+$, $B = \mathbb{D}^n_-$, so that $A \cap B = \mathbb{S}^{n-1}$ (really, we mean a little regular neighborhood around these subsets).



Then we get in the Mayer-Vietoris sequence

$$H_n(A\cap B) \longrightarrow \underset{=0\oplus 0}{H_n(A)\oplus H_n(B)} \longrightarrow H_n(X) \longrightarrow H_{n-1}(A\cap B)$$

So, there are exact sequences for all $n \geq 2$

$$0 \longrightarrow H_n(X) \longrightarrow H_{n-1}(A\cap B) \longrightarrow 0$$

so we get that $H_n(\mathbb{S}^n) \cong H_{n-1}(\mathbb{S}^{n-1})$ for all $n \geq 2$.

- Given n-manifolds M and N for $n \geq 2$, the connect sum $M \# N$ is the n-manifold obtained via picking $\mathbb{D}_1^n \subset M$ and $\mathbb{D}_2^n \subset N$, and

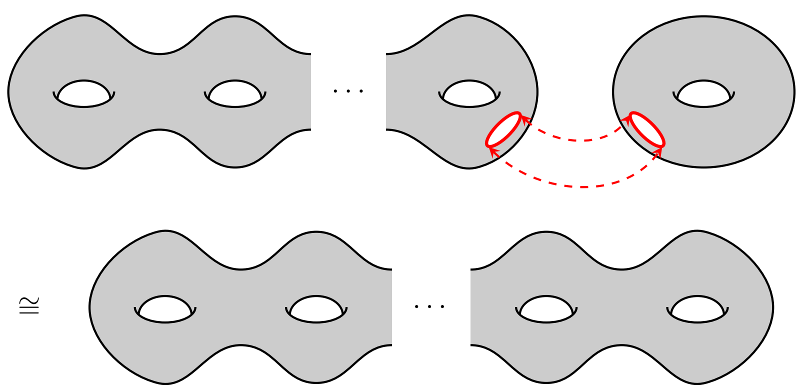
$$M \# N = \frac{(M - \mathrm{int}(\mathbb{D}_1^n)) \sqcup (N - \mathrm{int}(\mathbb{D}_2^N))}{\partial\mathbb{D}_1^n \sim \partial\mathbb{D}_2^n}$$

  We can apply Mayer-Vietoris to $M \# N$ by letting $X = M \# N$, $A = M - \mathrm{int}(\mathbb{D}_1^n)$, $B = N - \mathrm{int}(\mathbb{D}_2^n)$, so that $A \cap B = \partial\mathbb{D}_1^n = \partial\mathbb{D}_2^n \cong \mathbb{S}^{n-1}$. We get

$$H_n(\mathbb{S}^{n-1}) \longrightarrow H_n(M - \mathrm{int}(\mathbb{D}_1^n)) \oplus H_n(N - \mathrm{int}(\mathbb{D}_2^n)) \longrightarrow H_n(X) \longrightarrow H_{n-1}(M\#N)$$

  and we can compute e.g. $H_n(M - \mathrm{int}(\mathbb{D}_1^n))$ using Mayer-Vietoris in terms of homology of M.

  We can use this to get another computation of the homology of Σ_g, because $\Sigma_g \cong \Sigma_{g-1} \# \mathbb{T}^2$.



Note that $M \# \mathbb{S}^n \cong M$ for any n-manifold M.

Definition. A manifold M is prime when $M \cong A \# B$ for n-manifolds A, B, we must have $A \cong \mathbb{S}^n$ or $B \cong \mathbb{S}^n$.

Theorem (Kneser, 1920's)**.** *Every closed 3-manifold M can be expressed as a connect sum of prime closed 3-manifolds,*

$$M \cong M_1 \# \cdots \# M_r,$$

and this is unique up to ordering, i.e. if $M \cong N_1 \# \cdots \# N_s$, then $r = s$ and we can permute the indices so that $M_i \cong N_i$.

Theorem (Schoenflies)**.** $\mathbb{S}^2 - \mathbb{S}^1$ *has 2 components, each homeomorphic to* $\mathrm{int}(\mathbb{D}^2)$.

Lecture 17 (2012-11-07)

Cohomology

Fix an abelian group G, for example $G = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}/n\mathbb{Z}$.

For any abelian group A, $G^* = \mathrm{Hom}(A, G)$ is an abelian group under pointwise addition. For any homomorphism $\psi : A \to A'$, there is a dual homomorphism $\psi^* : (A')^* \to A^*$ defined by $\varphi \mapsto \varphi \circ \psi$, and for any $f : A \to A'$ and $g : A' \to A''$, we have $(g \circ f)^* = f^* \circ g^*$. Lastly, $(\mathrm{id}_A)^* = \mathrm{id}_{A^*}$.

Thus, we have proved that $\mathrm{Hom}(-, G)$ is a contravariant functor

$$\left\{ \begin{array}{c} \text{abelian groups} \\ \text{and homomorphisms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{abelian groups} \\ \text{and homomorphisms} \end{array} \right\}$$

which acts on objects by $A \mapsto A^*$, and acts on morphisms by $(f : A \to B) \mapsto (f^* : B^* \to A^*)$.

Given a chain complex $\mathcal{C} = \{C_n, \partial_n\}$ of free abelian groups, its nth homology $H_n(\mathcal{C})$ measures how far the complex is from being exact at C_n.

We will now form a new chain complex $\mathcal{C}^* = \{C_n^*, \delta_n\}$, where the δ_n are just the duals of the ∂_n. Specifically, looking at $\partial_n : C_n \to C_{n-1}$, we set $\delta_{n-1} = (\partial_n)^* : C_{n-1}^* \to C_n^*$. We'd like to take the homology of $\mathcal{C}^*$, but it's going the wrong way:

$$\cdots \xleftarrow{\delta_2} C_2 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_0} C_0 \longleftarrow 0$$

Note that $(\delta \circ \delta) = (\partial^* \circ \partial^*) = 0$, so we will call the δ's coboundary maps, and we call $\mathcal{C}^*$ a cochain complex. The group C_n^* is called the group of n-cochains.

Definition. Given a chain complex $\mathcal{C} = \{C_n, \partial_n\}$ and abelian group G, the associated cochain complex $\mathcal{C}^* = \{C_n^*, \delta_n\}$ is defined as above, and the cohomology of $\mathcal{C}$ with coefficients in G is defined to be

$$H^n(\mathcal{C}; G) = \frac{\ker(\delta_n)}{\mathrm{im}(\delta_{n-1})}.$$

Given a chain map $f : \mathcal{C} \to \mathcal{C}'$, we get an induced map on cohomology in the other direction, $f_n^* : H^n(\mathcal{C}'; G) \to H^n(\mathcal{C}; G)$. This is because

$$f \circ \partial = \partial \circ f \quad \Longrightarrow \quad \delta \circ f^* = \partial^* \circ f^* = f^* \circ \partial^* = f^* \circ \delta.$$

This also implies that if f is chain homotopic to g, then $f^* = g^*$.

Theorem (FTHA, cohomology version)**.** *Given a short exact sequence of chain complexes*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

there is a long exact sequence

$$\cdots \longleftarrow H^{n+1}(\mathcal{C}) \longleftarrow H^n(\mathcal{A}) \longleftarrow H^n(\mathcal{B}) \longleftarrow H^n(\mathcal{C}) \longleftarrow \cdots$$

Next class we will prove the universal coefficient theorem, which will allow us to compare homology with cohomology. Let's do an example first though. Fix $G = \mathbb{Z}$, and consider this chain complex:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

The dual of the zero map is obviously the zero map. Note that $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, with the identity map $\mathrm{id}_{\mathbb{Z}}$ as the generator. It is easy to see that $(\times 2) : \mathbb{Z} \to \mathbb{Z}$ induces the map from $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ to $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ which is also multiplication by 2.

Thus, corresponding cochain complex is

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

Therefore, the homology and cohomology are

$$H_i(\mathcal{C}) = \left\{ \begin{array}{ccc} \mathbb{Z} & i = 0 & \mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & i = 1 & 0 \\ 0 & i = 2 & \mathbb{Z}/2\mathbb{Z} \end{array} \right\} = H^i(\mathcal{C}, \mathbb{Z})$$

So in general, how can we compare $H^*(\mathcal{C}; G)$ with $H_*(\mathcal{C})$?

We claim that there is a homomorphism $\Psi : H^n(\mathcal{C}; G) \to \mathrm{Hom}(H_n(\mathcal{C}); G)$. We construct it as follows. Given a $\varphi \in Z_n^* \subset C_n^* = \mathrm{Hom}(C_n, G)$, we have that $\delta\varphi = 0 \in C_{n+1}^*$. Thus, for any $\sigma \in C_{n+1}$, we have that $(\delta\varphi)(\sigma) = (\varphi \circ \partial)(\sigma) = \varphi(\partial\sigma) = 0$, so $\varphi(B_n) = 0$. Therefore, $\varphi : C_n \to G$ factors through a map $\widetilde{\varphi} : C_n/B_n \to G$, and then restricting to Z_n, we get a map $\widetilde{\varphi}|_{Z_n} : Z_n/B_n \to G$, i.e. a map from $H_n(\mathcal{C})$ to G, which is exactly the kind of object we are claiming Ψ sends φ to.

You should check on your own that Ψ is well-defined, and that in general Ψ may have a kernel.

Lecture 18 (2012-11-09)

The Universal Coefficient Theorem

Fix an abelian group G, and let $\mathcal{C} = \{C_n, \partial_n\}$ be a chain complex. As we discussed last time, this gives us a cochain complex $\mathcal{C}^* = \{C_n^*, \delta^n = \partial_{n-1}^*\}$, and then the cohomology groups $H^n(\mathcal{C}; G)$ with coefficients in G.

Last time, we defined a map $\Psi : H^n(\mathcal{C}; G) \to \mathrm{Hom}(H_n(\mathcal{C}), G)$. It is easy to show that Ψ is onto, so we get a short exact sequence

$$1 \longrightarrow \ker(\Psi) \longrightarrow H^n(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}), G) \longrightarrow 1$$

We want to compute $\ker(\Psi)$, since it represents the difference between homology and cohomology.

You should read in Hatcher about the functor $\mathrm{Ext}(-, G)$, which goes from Ab to Ab, and also read about free resolutions. The key step will be showing that $\ker(\Psi) = \mathrm{Ext}(H_{n-1}(\mathcal{C}), G)$. This then imples the universal coefficient theorem, because we get a short exact sequence

$$1 \longrightarrow \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \longrightarrow H^n(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}), G) \longrightarrow 1$$

The following proposition explains how to compute $\mathrm{Ext}(-, G)$.

Proposition.

1. $\mathrm{Ext}(A \oplus B, G) = \mathrm{Ext}(A, G) \oplus \mathrm{Ext}(B, G)$.
2. $\mathrm{Ext}(A, G) = 0$ *if A is a free abelian group.*
3. $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$, *where $nG = \{ng \mid g \in G\}$.*

In fact, when $G = \mathbb{Z}$, we have

$$\mathrm{Ext}(H_{n-1}(\mathcal{C}, G)) \cong \frac{H_n(\mathcal{C})}{\mathrm{torsion}(H_n(\mathcal{C}))} \oplus \mathrm{torsion}(H_{n-1}(\mathcal{C})).$$

Remark. The functor $\mathrm{Ext}(A, G)$ is called that because it has to do with extensions of G by A, i.e. ways of picking a group Γ so that

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

Thus, cohomology is a key tool for the classification of groups.

Cohomology of Spaces

For any topological space X (respectively, any Δ-complex, CW-complex), we define

$$\begin{aligned} H^n(X; G) &:= H^n(\{C_n^{\mathrm{sing}}(X)\}) \\ H^n_{\mathrm{CW}}(X; G) &:= H^n(\{C_n^{\mathrm{CW}}(X)\}) \\ H^n_{\Delta}(X; G) &:= H^n(\{C_n^{\Delta}(X)\}) \end{aligned}$$

Then because the universal coefficient theorem is natural in $\mathcal{C}$, and the isomorphisms between the various homology theories are natural, we can conclude that

$$H^n(X; G) \cong H^n_{\Delta}(X; G) \cong H^n_{\mathrm{CW}}(X; G)$$

when they are all defined (i.e. when X is a Δ-complex, CW-complex). Moreover, naturality together with the corresponding fact about homology implies that $f \sim g$ we have $f^* = g^*$ (so that H^i is a homotopy functor), and we even get the long exact sequence of a pair, excision, and Mayer-Vietoris for cohomology, all from naturality.

Cup Product

Fix a ring R (e.g. $R = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/d\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}_p$, $\ldots$). Let X be a space. Considering the underlying additive abelian group of R, we get the abelian groups $H^n(X;R)$ for all $n \geq 0$, and we let

$$H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

which is a (graded) abelian group. Our goal is to make $H^*(X;R)$ into a ring. This additional ring structure on cohomology will let us distinguish $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and $\mathbb{T}^2$, even though

$$H_i(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) = \left\{ \begin{array}{cc} \mathbb{Z} & i=0 \\ \mathbb{Z}^2 & i=1 \\ \mathbb{Z} & i=2 \end{array} \right\} = H_i(\mathbb{T}^2).$$

For $a \in C^i(X;R) = \mathrm{Hom}(C_i(X);R)$ and $b \in C^j(X;R)$, for any $(\sigma : [v_0 v_1 \cdots v_{i+j}] \to X) \in C_{i+j}(X)$ we define $a \smile b \in C^{i+j}(X;R)$ by

$$(a \smile b)(\sigma) = a(\sigma|_{[v_0 v_1 \cdots v_i]}) \underset{\substack{\uparrow \\ \text{mult. in } R}}{\cdot} b(\sigma|_{[v_i \cdots v_{i+j}]}).$$

We have the "graded product rule for δ":

$$\delta(a \smile b) = \delta a \smile b + (-1)^i a \smile \delta b$$

This implies that if $\delta a = \delta b = 0$, then $\delta(a \smile b) = 0$ and $\delta a \smile b = a \smile \delta b = 0$, so that the cup product map

$$\smile : C^i(X;R) \times C^j(X;R) \to C^{i+j}(X;R)$$

induces a map in homology

$$\smile : H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R),$$

defined by $[a] \smile [b] = [a \smile b]$. This is a bilinear map, so passing to the tensor product and putting all degrees together, we get a homomorphism

$$\smile : H^*(X;R) \otimes H^*(X;R) \to H^*(X;R),$$

In summary, $H^*(X;R)$ is a graded commutative ring $a \smile b = (-1)^{ij} b \smile a$, where addition is just addition of cocycles, and multiplication is $\smile$. For any map $f : X \to Y$, we get an induced ring homomorphism $f^*(a \smile b) = f^*(a) \smile f^*(b)$.

The additive and multiplicative identity are the constant functions $0, 1 \in H^0(X;R) = \mathrm{Hom}(C_0(X);R)$.

Lecture 19 (2012-11-12)

Today we'll look at the cup product structure on the cohomology of a few spaces.

Let's start by considering $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ vs. $\mathbb{T}^2$.

Both of these spaces have CW-complex structures with one 0-cell, two 1-cells, and one 2-cell. The difference is that the 2-cell was attached via a homotopically trivial map in $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$, while it was attached via the the path $aba^{-1}b^{-1}$ in $\mathbb{T}^2$. While homotopically these are different, on the level of homology they cannot be distinguished because H_1 is the abelianization of the fundamental group. Surprisingly, the difference can be detected in the cup product structure.

The homology of both of these spaces is

$$H_*(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) \cong H_*(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \text{in dim } 0 \\ \mathbb{Z}^2 & \text{in dim } 1 \\ \mathbb{Z} & \text{in dim } 2 \\ 0 & \text{in higher dim} \end{cases}$$

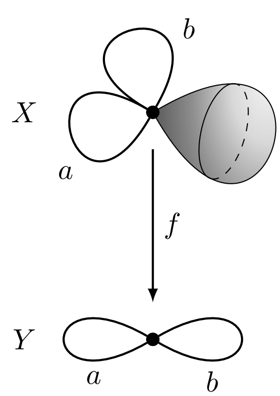
Because all of these groups are free, the Ext groups in the universal coefficient theorem are trivial, and therefore the cohomology of both spaces is isomorphic to the homology.

What about the cup product structure on H^*? The only interesting case is in H^1. We have that $H^1 = \mathbb{Z}^2 = \mathrm{Hom}(H_1, \mathbb{Z})$ is generated by two 1-cocycles α and β, where $\alpha(a) = 1$ and $\alpha(b) = 0$, and $\beta(a) = 0$ and $\beta(b) = 1$.

By the anti-commutativity of the cup product in odd dimensions, for either space we have $\alpha \smile \alpha = -(\alpha \smile \alpha)$ and $\beta \smile \beta = -(\beta \smile \beta)$, and because H^2 has no 2-torsion, this implies $\alpha \smile \alpha = \beta \smile \beta = 0$ for either space.

What about $\alpha \smile \beta$?

We map $X = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ down to $Y = \mathbb{S}^1 \vee \mathbb{S}^1$ by collapsing the 2-cell to a point:



We have that

$$H^*(Y) = \begin{cases} \mathbb{Z} & \text{in dim } 0 \\ \mathbb{Z}^2 & \text{in dim } 1 \\ 0 & \text{in higher dim} \end{cases}$$

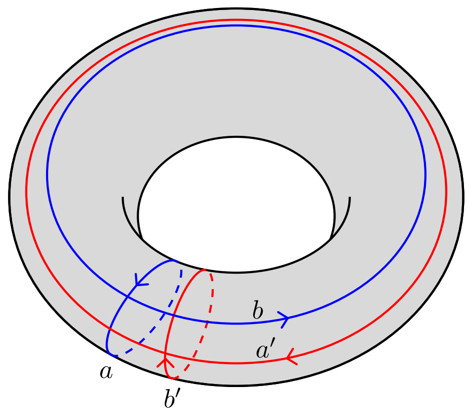
Let $\gamma, \delta \in H^1(Y)$ be generators of $H^1(Y) \cong \mathbb{Z}^2$. Clearly, $\gamma \smile \delta = 0$ because Y has no cohomology in dimension 2. Because we have

$$f^*(\gamma)(a) = \gamma(f_*(a)) = \alpha(a), \qquad f^*(\gamma)(b) = \gamma(f_*(b)) = \alpha(b)$$

and similarly with δ and β, we can conclude that

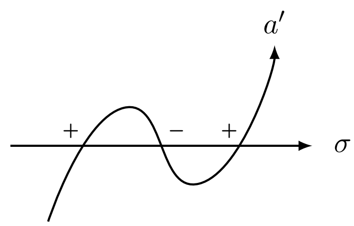
$$0 = f^*(0) = f^*(\gamma \smile \delta) = f^*(\gamma) \smile f^*(\delta) = \alpha \smile \beta.$$

Now let's look at $\mathbb{T}^2$. The lines a and b represent homology classes which will generate $H^1(\mathbb{T}^2)$, and we will also draw lines a' and b' which will let us define the cocycles which are "dual" to $[a]$ and $[b]$.

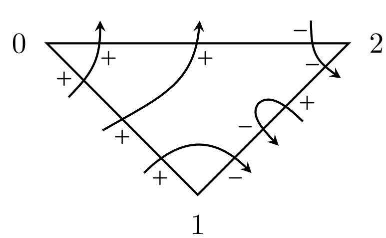


We define cochains α, β by

$$\alpha(\sigma) = \text{(signed) \# of intersections of } a',$$
$$\beta(\sigma) = \text{(signed) \# of intersections of } b'.$$



We claim that these cochains are actually cocycles. Looking at the intersections of a' with the faces of a 2-simplex,

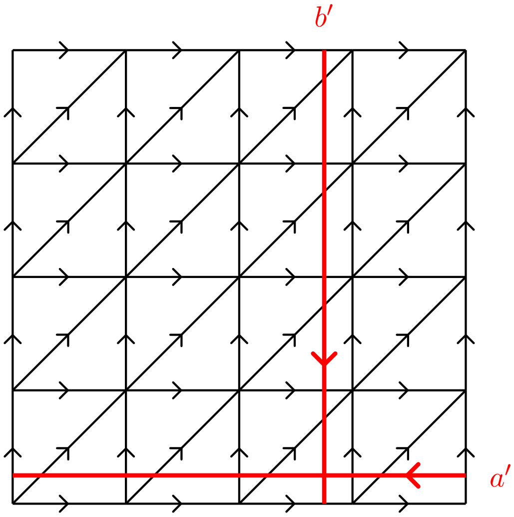


and adding up all the signs on $[01] + [12] - [02]$, by the classification of 1-manifolds we know there are an even number of intersections and the sum must cancel.

Thus, α and β represent classes $[\alpha], [\beta] \in H^1(\mathbb{T}^2)$. We have

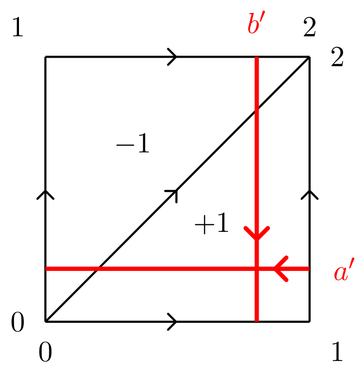
$$[\alpha]([a]) = 1, \quad [\alpha]([b]) = 0, \quad [\beta]([a]) = 0, \quad [\beta]([b]) = 1.$$

Triangulating the torus, and drawing a' and b',



Recall that, by definition, we compute $\alpha \smile \beta$ by having α eat the front face of a triangle, and β eat the back face, i.e. $\alpha(T_{01})\beta(T_{12})$. We will compute $(\alpha \smile \beta)[\mathbb{T}^2]$ by adding up the contribution from all of the triangles in our triangulation.

However, in all but two triangles, there is no contribution because either a' or b' does not pass through at all, so that α or β vanish. The only interesting piece is

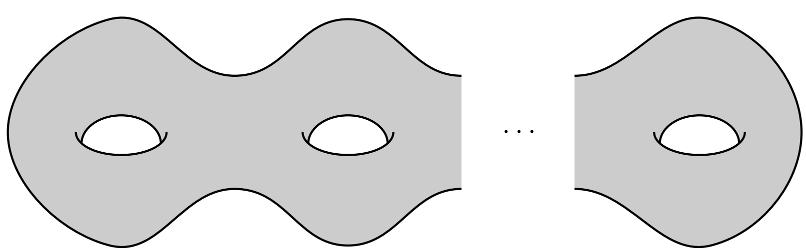


We see that

$$([\alpha] \smile [\beta])([\mathbb{T}^2]) = (-1) \cdot [(+1)(-1)] + (+1) \cdot [(0)(0)] = 1,$$

which isn't 0, so the cup product structures on $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and $\mathbb{T}^2$ are different.

Now let's do a similar calculation for S_g, the closed oriented surface of genus g.



We know that

$$H_*(S_g) = \begin{cases} \mathbb{Z} & \text{in dim } 0 \\ \mathbb{Z}^{2g} & \text{in dim } 1 \\ \mathbb{Z} & \text{in dim } 2 \\ 0 & \text{in higher dim} \end{cases}$$

Taking inspiration from what we did with the torus, we define analogous curves $a'_1, \dots, a'_g$ and $b'_1, \dots, b'_g$ on S_g, and then we define cocycles α_i and β_i which count the signed number of intersections with a'_i or b'_i, respectively. Thus

$$\begin{aligned} &\alpha_i(a_i) = 1 && \beta_i(b_i) = 1 \\ &\alpha_i(b_j) = 0 \text{ for all } j && \beta_i(a_j) = 0 \text{ for all } j \\ &\alpha_i(a_j) = 0 \text{ for all } j \neq i && \beta_i(b_j) = 0 \text{ for all } j \neq i \end{aligned}$$

To compute cup products, we choose a triangulation, so that the fundamental class $[S_g]$ can be represented by a sum of triangles. Then

$$([\alpha_i] \smile [\beta_i])([S_g]) = 1, \quad [\alpha_i] \smile [\beta_j] = 0, \quad [\beta_i] \smile [\beta_j] = 0, \quad [\alpha_i] \smile [\beta_j] = 0 \text{ for } i \neq j.$$

This is a non-degenerate, alternating (i.e. antisymmetric) pairing on H^1 (such a form is called symplectic). This is a general phenomenon - in even-dimensional manifolds, the cup product on middle-dimensional cohomology gives a nice pairing to the top-dimensional cohomology (which is just $\mathbb{Z}$ when it is orientable). If the dimension of the manifold is 0 mod 4, then the middle dimension is even, and we get a symmetric form; if the dimension of the manifold is 2 mod 4, then the middle dimension is odd, and we get an anti-symmetric form.

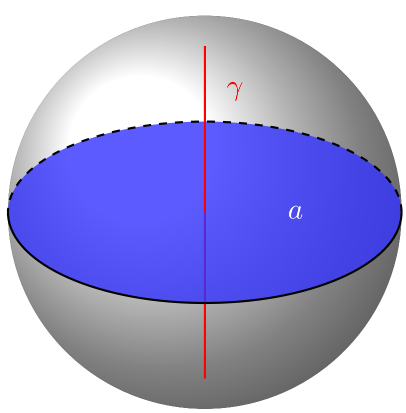
Let's finish with $\mathbb{RP}^n$. So far, we've been working with 2-dimensional manifolds, so looking for things that 1-dimensional submanifolds intersect with gave us other 1-dimensional submanifolds. In general, we need to look at 1-codimensional submanifolds.

Because $\mathbb{RP}^n$ is non-orientable, we can't talk about the signed number of intersections, but we can work in $\mathbb{Z}/2\mathbb{Z}$.

Viewing $\mathbb{RP}^n$ as $\mathbb{D}^n/(\pm 1 \text{ on } \partial\mathbb{D}^n)$, there are n different copies of $\mathbb{RP}^{n-1}$ in $\mathbb{RP}^n$, each perpendicular to one of the coordinate axes. Choose one, which we will call a, and define α by

$$\alpha(\sigma) = \# \text{ of intersections of } \sigma \text{ with } a \pmod 2$$

In particular, $\alpha(\gamma) = 1 \neq 0 \pmod 2$, where γ is the coordinate axis perpendicular to the chosen copy of $\mathbb{RP}^{n-1}$.



We know that

$$H_*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise,} \end{cases}$$

and because $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_*, \mathbb{Z}/2\mathbb{Z})$, the cohomology is the same as the homology. Because $H^1(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is a 1-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$, no matter which of the n copies of $\mathbb{RP}^{n-1}$ we chose to define α with, we would get the same cohomology class in H^1 (the $\alpha_1, \alpha_2, \ldots$ are all different as cocycles, though).

Thus,

$$\big([\alpha_1] \smile [\alpha_2] \smile \cdots \smile [\alpha_n]\big)[\mathbb{RP}^n] = \big([\alpha] \smile [\alpha] \smile \cdots \smile [\alpha]\big)[\mathbb{RP}^n] = 1,$$

and therefore

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\alpha]/[\alpha]^{n+1}$$

where α is a degree 1 element. In fact,

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\alpha].$$

Lecture 20 (2012-11-14)

Kunneth Theorem

Our goal is to compute $H^*(X \times Y; R)$ in terms of $H^*(X; R)$ and $H^*(Y; R)$. Recall that the product $X \times Y$ comes with natural projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\ X & & Y \end{tikzcd}$$

Definition. The $\times$ product on cohomology, $\times : H^*(X; R) \times H^*(Y; R) \to H^*(X \times Y; R)$, is defined by sending $a \times b$ to $P_1^*(a) \smile P_2^*(b)$.

The map $\times$ is bilinear, and $\times$ induces a homomorphism of R-modules

$$\Psi : H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R).$$

In fact, we can make this into a homomorphism of graded rings by *defining*

$$(a \otimes b) \cdot (c \otimes d) := (-1)^{|b||c|}(ac \otimes bd)$$

where $|b|$ and $|c|$ are the degrees of b and c.

Theorem (Kunneth)**.** *If X and Y are CW-complexes, and if $H^*(Y; R)$ is a free R-module, then Ψ is an isomorphism.*

Example. The cohomology ring of $\mathbb{T}^n$ over $\mathbb{Z}$ is

$$H^*(\mathbb{T}^n, \mathbb{Z}) \cong \Lambda^*[x_1, \ldots, x_n] = \bigoplus_{i=0}^{n} \Lambda^i[x_1, \ldots, x_n]$$

where $\Lambda^i[x_1, \ldots, x_n]$ is the free abelian group on

$$\{x_{i_1} \wedge \cdots \wedge x_{i_k} \mid i_1 < \cdots < i_k\},$$

so that $\operatorname{rank}(\Lambda^i[x_1, \ldots, x_n]) = \binom{n}{i}$.

Proof. We induct using Kunneth:

$$H^*(\mathbb{T}^n, \mathbb{Z}) \cong H^*(\mathbb{T}^{n-1} \times \mathbb{S}^1; \mathbb{Z}) = H^*(\mathbb{T}^{n-1}) \otimes H^*(\mathbb{S}^1).$$

For example,

$$\begin{aligned} H^3(\mathbb{T}^4) &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \\ &= [H^3(\mathbb{T}^3) \otimes H^0(\mathbb{S}^1)] \oplus [H^2(\mathbb{T}^3) \otimes H^1(\mathbb{S}^1)] \oplus [H^1(\mathbb{T}^3) \otimes H^2(\mathbb{S}^1)] \oplus [H^0(\mathbb{T}^3) \otimes H^3(\mathbb{S}^1)] \\ &= [\mathbb{Z} \otimes \mathbb{Z}] \oplus [\Lambda^2\mathbb{Z}^3 \otimes \mathbb{Z}] \oplus 0 \oplus 0 \\ &= \mathbb{Z} \oplus \Lambda^2\mathbb{Z}^3. \end{aligned}$$

□

When Ψ is an isomorphism, we have

$$H^n(X \times Y; \mathbb{Z}) = \bigoplus_{p+q=n} [H^p(X) \otimes H^q(Y)].$$

In Hodge theory, we see that there is a decomposition like this even when we're not looking at a product manifold.

Prologue to Poincaré duality

Definition. A second countable, Hausdorff space M is an n-dimensional manifold (possibly wth boundary) if for any $m \in M$, there is a neighborhood U_m such that either $U_m \cong \mathbb{R}^n \cong B^n$ (the open ball in $\mathbb{R}^n$), or $U_m \cong \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i \geq 0\}$ (the closed upper half space). Key fact:

$$\{m \in M \text{ for which the latter holds}\} \cong \text{ an } (n-1)\text{-dimensional manifold } \partial M.$$

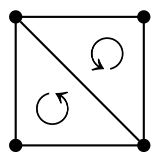
Definition. A compact manifold M is called triangulable if there is a simplicial complex $K \cong M$. Most of the manifolds which occur in real life are triangulable, and we might as well assume that all manifolds we'll use in class will be triangulable.

Definition. An orientation on a simplex $[v_0 \cdots v_n]$ is a choice of one of the two equivalence classes of $(n+1)$-tuples $(v_{i_0} \cdots v_{i_n})$, where two $(n+1)$-tuples are equivalent when the permutation sending one to the other is an even permutation.

Definition. M is orientable if one of the following equivalent statements holds.

1. There is a triangulation on M which can be oriented compatibly.
2. All triangulations on M can be oriented compatibly.

We know that under any reasonable definition, a disk ought to be orientable; giving the disk a triangulation and assigning the same orientation to each simplex, we get



This motivates the definition of compatible orientation to be a choice of orientation on each n-simplex σ_i such that if $\sigma_i \cap \sigma_j$ is a (codimension 1) face of each, then σ_i and σ_j induce opposite orientations on that face.

Exercise.

1. Prove that Σ_g is orientable for all $g \geq 1$.
2. Prove that the Mobius strip and Klein bottle are not.
3. How about $\mathbb{R}\mathrm{P}^n$?

Theorem (Poincaré duality)**.** *Let M be a closed (i.e. compact, with no boundary), orientable manifold of dimension n. Then*

$$H^i(M; R) \cong H_{n-i}(M; R).$$

Corollary. *If M is a closed manifold of odd dimension, then $\chi(M) = 0$.*

Proof of corollary. Let $b_i = \dim(H_i(M; \mathbb{R}))$. Then we know $b_i = \dim(H^i(M; \mathbb{R}))$ by the universal coefficient theorem, but by Poincaré duality b_i is also equal to $\dim(H^{n-i}(M; \mathbb{R}))$. Thus

$$\chi(M) = b_0 - b_1 + b_2 - \cdots - b_{2k+1}$$

cancels out, because $b_0 = b_{2k+1}$, $b_1 = b_{2k}$, etc. $\square$

Lecture 21 (2012-11-16)

Another Corollary of Poincaré Duality

Here is a useful algebraic fact: any non-degenerate skew-symmetric bilinear form on $\mathbb{Z}^n$ is, after changing basis, of the form

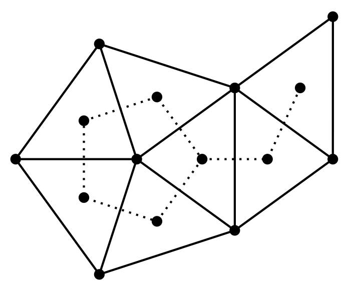
$$\begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, n must be even. This fact, together with Poincaré duality, imply

Corollary. *Let M be a closed, oriented, $(4k+2)$-dimensional manifold. Then because we have a non-degenerate pairing $H^{n-i}(M) \times H^i(M) \to \mathbb{Z}$, $\operatorname{rank}(H^i(M,\mathbb{Z})) = \chi(M)$ must be even.*

Continued Prologue to Poincaré duality

If M be a closed, orientable n-manifold, then $H_i(M;\mathbb{Q}) \cong H^{n-i}(M;\mathbb{Q})$. Poincaré's idea about this was as follows: in a Δ-complex, we compute H_i using $C_i^{\Delta}(M)$. We can build a "dual" CW-complex (it won't be a Δ-complex in general, we may get a polygonal complex) by putting a 0-cell in the center of every n-simplex, adding a 1-cell connecting two n-simplices if they meet on a codimension 1 subface, etc.



Thus, $\partial \longleftrightarrow \delta$ is an isomorphism

$$C_i^{\Delta}(X) \cong C_{n-i}^{\mathrm{cw}}(X').$$

Thus, $\{C_i^{\Delta}(X), \partial\}$ computes $H_i(X) \cong H_i(M)$, and $\{C_{n-1}^{\Delta}(X), \delta\}$ computes $H^{n-i}(X') \cong H^{n-i}(M)$.

Classification of Surfaces

Theorem 0.1. *Let M be a closed 2-manifold. Then M is homeomorphic to one of:*

- Σ_g *where* $g \geq 0$ *(when M orientable), or*
- $\Sigma_g \# \mathbb{RP}^2$ *(when M non-orientable).*

Theorem 0.2 (2-dimensional Poincaré conjecture)**.** *The following are equivalent:*

1. $\chi(M) = 2$
2. $M \cong \mathbb{S}^2$
3. *Every loop $\gamma \subset M$ (i.e. embedding $\gamma : \mathbb{S}^1 \to M$) separates M into 2 components,*

Statement 2 implies statement 3 by the Jordan separation theorem, and we already know that statement 2 implies statement 1.

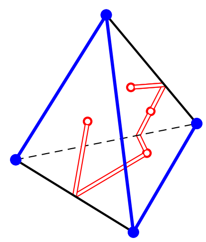
We will be assuming the following classical result:

Theorem (Rado, 1920's)**.** *Every surface is triangulable.*

Proof of Theorem 2. We will show that statement 1 implies statement 2. Let K be a triangulation of M. First, pick a maximal tree T in $K^{(1)}$. Note that $T \supset K^{(0)}$.

Now build a connected graph $\Gamma \subset K$ as follows: $V(\Gamma)$ consists of the 2-simplices of K, and $E(\Gamma)$ consists of $K^{(1)} - T^{(0)}$.

Here is an example on a tetrahedron:



We have

$$\begin{aligned}
\chi(M) &= V(K) - E(K) + F(K) \\
&= V(T) - (E(T) + E(\Gamma)) + V(\Gamma) \\
&= [V(T) - E(T)] + [V(\Gamma) - E(\Gamma)] \\
&= \underbrace{\chi(T)}_{\leq 1} + \underbrace{\chi(\Gamma)}_{\leq 1} \\
&\leq 2,
\end{aligned}$$

where we have used the following lemma.

Lemma. *Let Ω be a finite connected graph. Then $\chi(\Omega) \leq 1$ with equality if and only if Ω is a tree.*

Proof of lemma. Trivial application of induction (check for yourself). $\square$

Thus, we have that $\chi(M) \leq 2$, with equality if and only if $\chi(T) = \chi(\Gamma) = 1$, which is the case if and only if Γ is a tree. But if $T \subset M$ is a tree, there is a neighborhood of T homeomorphic to $\mathbb{D}^2$. Being very careful, we can thicken Γ and T to neighborhoods each homeomorphic to $\mathbb{D}^2$, and such that

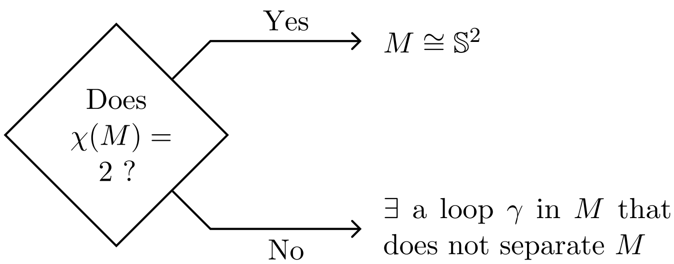
$$\overline{\text{nbhd}(T)} \cap \overline{\text{nbhd}(\Gamma)} \cong \mathbb{S}^1 = \partial \mathbb{D}^2_1 = \partial \mathbb{D}^2_2.$$

This implies that $M \cong \mathbb{S}^2$.

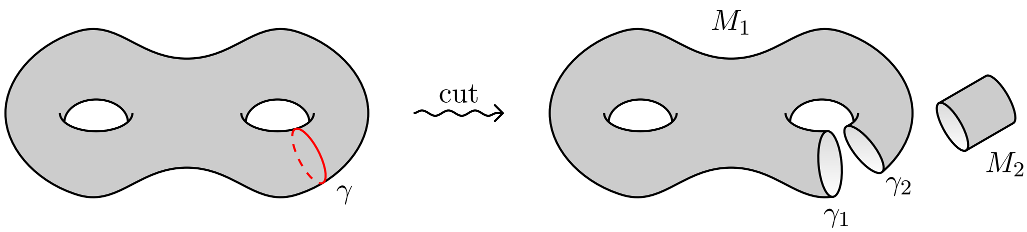
Now, note that statement 3 implies that Γ is a tree: if it weren't, it would have a loop γ, and γ would have to separate M; but then it would separate two vertices of $K^{(0)}$, which would contradict that $\Gamma \cap T = \varnothing$.

Thus, statement 3 implies statement 1. $\square$

Proof of Theorem 1. We are given M, which has $\chi(M) \leq 2$ by the above.



A neighborhood of γ is homeomorphic to either $\mathbb{S}^1 \times [0,1]$ or the Möbius strip (the latter is only possible if M is not orientable).



Let M' be M_1 with γ_1 and γ_2 filled in by disks. Thus, M' is a new closed, connected surface, and

$$\chi(M) = \chi(M_1) + \chi(M_2) - \chi(\mathbb{S}_1 \sqcup \mathbb{S}_1).$$

It is easy to check that $\chi(M') > \chi(M)$; in fact, $\chi(M') = \chi(M) + 2$. $\square$

Lecture 22 (2012-11-19)

The Mayer-Vietoris Argument

Definition. An open cover $\{U_i\}$ of an n-manifold M is a good cover if each $U_i \cong \mathbb{R}^n$ and for all $i_1, \ldots, i_r$, $U_{i_1} \cap \cdots \cap U_{i_r} \cong \mathbb{R}^n$ or is empty.

Theorem. *Every smooth manifold M has a good cover. Of course, if M compact, then we can choose the good cover to be finite.*

Proof. First, we will prove that any smooth manifold M admits a Riemannian metric. Take a cover of M by coordinate patches U_i, and on each coordinate patch, let $\langle \cdot\,, \cdot \rangle_i$ be the pullback of the standard Riemannian metric on $\mathbb{R}^n$. Let P_i be a partition of unity subordinate to the cover. Then $\langle \cdot\,, \cdot \rangle = \sum P_i \langle \cdot\,, \cdot \rangle_i$ is a Riemannian metric on M.

Now we appeal to a theorem of Whitehead (or Alexander): if M is any Riemannian manifold, then every $m \in M$ has a convex neighborhood U_m, where convex means that there is a unique geodesic in U_m between any two points. Because the intersection of convex sets is convex, and convex $\cong \mathbb{R}^n$, we are done. $\square$

Remark. Good covers are cofinal in the set of covers ordered by refinement.

Theorem. *If a manifold M has a finite good cover, then the ith homology $H_i(M)$ of M is finitely generated for all $i \geq 0$.*

Proof. Mayer-Vietoris implies that for open $U, V \subseteq M$, we have an exact sequence

$$\cdots \longrightarrow H_i(U) \oplus H_i(V) \longrightarrow H_i(U \cup V) \longrightarrow H_{i-1}(U \cap V) \longrightarrow \cdots$$

Thus, by the rank-nullity theorem,

$$\dim(H_i(U \cup V)) \leq \dim(H_{i-1}(U \cap V)) + \dim(H_i(U)) + \dim(H_i(V)),$$

where dim here means the minimial number of generators.

If $U_1, \ldots, U_r$ is a good cover, then $U = U_1 \cup \cdots \cup U_{r-1}$ is an open submanifold of M, and we can let $V = U_r$. By induction on the size of a minimal open cover, we have that $H_i(U)$ and $H_i(V)$ are finitely generated, and thus we are done by our observation above.

The base case is just $M = \mathbb{R}^n$ which is trivial. $\square$

Künneth theorem via a Mayer-Vietoris argument

Theorem. *Let $H^*(X) = H^*(X; \mathbb{Q})$. Then for any manifolds M_1, M_2 with finite good covers (in fact we only need one of them to have this), we have*

$$H^*(M_1 \times M_2) \cong H^*(M_1) \otimes H^*(M_2).$$

Proof. We will show the map $\psi : H^*(M_1) \otimes H^*(M_2) \to H^*(M_1 \times M_2)$ defined by sending $a \otimes b$ to $a \times b$ is an isomorphism.

Let $U_1, \ldots, U_r$ be a good cover of M. We induct on r.

For $r = 1$, since $\mathbb{R}^n \times M_2 \simeq M_2$, we have $H^*(M_2) \cong H^*(\mathbb{R}^n \times M_2) \cong H^*(\mathbb{R}) \otimes H^*(M_2)$.

Now suppose that $\{U, V\}$ is a cover of M_1. Mayer-Vietoris implies that we have a long exact sequence

$$\cdots \longrightarrow H^i(U \cap V) \longrightarrow H^i(U) \oplus H^i(V) \longrightarrow H^i(U \cap V) \longrightarrow \cdots$$

Tensoring with $H^{n-i}(M_2)$ throughout (?), we get

$$\cdots \to H^i(U \cap V) \otimes H^{n-i}(M_2) \to (H^i(U) \oplus H^i(V)) \otimes H^{n-i}(M_2) \to H^i(U \cap V) \otimes H^{n-i}(M_2) \to \cdots$$

which is still exact because tensoring with a vector space preserves exactness.

Summing the above exact sequence over all i, we get an exact sequence

$$\cdots \longrightarrow \underbrace{\bigoplus_{i=1}^{n} [H^i(U \cup V) \otimes H^{n-i}(M-2)]}_{n\text{th degree of } H^*(U \cup V) \otimes H^*(M_2)} \longrightarrow \cdots$$

which ψ maps to

$$\cdots \longrightarrow H^n((U \cup V) \times M_2) \longrightarrow \cdots$$

Check that ψ forms a chain map between these two exact sequences, and use the 5 lemma. (?) $\square$

Orientation Revisited

Let $x \in \mathbb{R}^n$, and let B be a neighborhood of x. We have

$$\begin{aligned} H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}) &\underset{\substack{\uparrow \\ \text{excision}}}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n - \overline{B}) \\ &\cong \widetilde{H}_i(\mathbb{R}^n / (\mathbb{R}^n - \overline{B})) \\ &\cong \widetilde{H}_i(\overline{B}/\partial B) \\ &\cong \widetilde{H}_i(\mathbb{S}^n) \\ &\cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now let M be a manifold. For any $x \in M$, we can find a neighborhood B of x lying in a coordinate chart. By the same reasoning,

$$\begin{aligned} H_i(M, M - \{x_i\}) &\cong H_i(B, B - \{x\}) \\ &\cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

A choice of generator of $H_n(M, M - \{x\}) \cong \mathbb{Z}$ is called a local orientation at x.

Now let M be a triangulated, closed, and connected n-manifold for which there exists a compatible orientation on each simplex of the triangulation. Let $\{\sigma_i\}$ be the set of n-simplices, and let $\sigma = \sum \sigma_i \in C_n(M)$; this sum makes sense because M is compact, so there are finitely many terms.

Because M is a manifold without boundary, and the orientations on $\sigma_i \cap \sigma_j$ cancel, we have that $\partial\sigma = \sum \partial\sigma_i = 0$. Thus, $\sigma \in Z_n(M)$. But $C_{n+1}(M) = 0$ because M is n-dimensional; therefore, $[\sigma] \in H_n(M)$ is of infinite order. This homology class is called the fundamental class of M, and is denoted $[M]$.

Corollary. *The Klein bottle K is not orientable since $H_2(K) = 0$.*

Check for yourself that for any $x \in M$, the inclusion map $i : (M, \varnothing) \hookrightarrow (M, M - x)$ induces a map $i_* : H_n(M) \to H_n(M, M - x)$ that takes a generator to a generator.

Lecture 23 (2012-11-21)

Poincaré Duality

Definition. Let X be a space. The cap product is a pairing between certain homology groups and cohomology groups of X. For $k \geq \ell$, we define $\frown : C_k(X) \times C^\ell(X) \to C_{k-\ell}(X)$ by taking $\sigma \in C_k(X)$, a singular k-chain $\sigma : [v_0 \cdots v_k] \to X$, and $\phi \in C^\ell(X)$, a singular ℓ-cochain, and mapping them to

$$\sigma \frown \phi = \phi(\sigma|_{[v_0 \cdots v_\ell]})\sigma|_{[v_{\ell+1} \cdots v_k]}.$$

It is easy to check the following properties:

- $\frown$ is bilinear
- $\partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi - \sigma \frown \delta\phi)$
- $\frown(Z_k \times Z^\ell) \subseteq Z_{k-\ell}$, i.e. cycle $\frown$ cycle = cycle
- $\frown(B_k \times Z^\ell) \subseteq B_{k-\ell}$, i.e. boundary $\frown$ cycle = boundary

These facts imply that the cap product descends to a bilinear map $\frown : H_k(X) \times H^\ell(X) \to H_{k-\ell}(X)$.

Theorem (Poincaré Duality)**.** *Let M be a closed, oriented n-manifold. Then for any $0 \leq i \leq n$, the map $D_M : H^i(M) \to H_{n-i}(M)$ is an isomorphism, where D_M is defined by*

$$D_M([\phi]) = [M] \frown [\phi].$$

Corollary. *The top homology group $H_n(M)$ is isomorphic to $\mathbb{Z}$, and $[M]$ is a generator.*

Proof idea. We want to use a Mayer-Vietoris argument, but there's an immediate problem: the base case is false!

$$H_n(\mathbb{R}^n) = 0 \ncong \mathbb{Z} = H_0(\mathbb{R}^n).$$

To overcome this, we define $H^n_c(M)$, cohomology with compact support, which will satisfy

$$H^n_c(M) \cong H^n(M) \text{ for } M \text{ compact}$$
$$H^i_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we prove, using a Mayer-Vietoris argument, for all (not-necessarily-compact) connected, oriented manifolds Y without boundary that the map

$$D_Y : H^i_c(Y) \to H_{n-i}(Y)$$

is an isomorphism, and then finally, extend $\frown$ to the non-compact case. $\square$

Cohomology with Compact Support

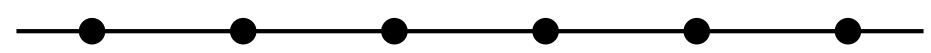
Let X be a locally finite Δ-complex. Define $C^i_c(X;R)$, the i-cochains with compact support, to be

$$C^i_c(X;R) := \{\phi \mid \phi = 0 \text{ outside a finite \# of simplices}\} \subseteq C^i(X;R) := \mathrm{Hom}(C_i(X), R).$$

Clearly, $\delta(C^i_c) \subseteq C^{i+1}_c$, so that $\zeta = \{C^i_c(X), \delta\}$ is a cochain complex. We then define the cohomology of X with compact support to be $H^i_c(X) := H^i(\zeta)$.

Note that H^i_c is only a (contravariant) functor when considering **proper** maps between spaces.

Example. Let's consider $X = \mathbb{R}$. We give it the following triangulation:



Then $C^0(\mathbb{R};\mathbb{Z})$ just consists of the functions on $\mathbb{R}$, and given $\phi \in C^0(\mathbb{R};\mathbb{Z})$, we have $\delta\phi = 0$ only if $\phi(v) = \phi(w)$ for all $w, v \in \mathbb{R}$, i.e. ϕ is constant. Therefore, if $\phi \in C_c^0(\mathbb{R})$, we must have $\phi = 0$. Thus $Z_c^0(\mathbb{R}) = 0$, and thus $H_c^0(\mathbb{R}) = 0$.

Now we claim that $H_c^1(\mathbb{R};\mathbb{Z}) \cong \mathbb{Z}$. Let $\Sigma : C_c^1(\mathbb{R}) \to \mathbb{Z}$ be the map sending ϕ to $\sum_{e \in X^{(1)}} \phi(e)$. Then for any $\psi \in C_c^0(\mathbb{R})$, we have

$$\delta\psi[i, i+1] = \psi(i+1) - \psi(i),$$

so that $\Sigma(\delta\psi) = 0$. Therefore, Σ induces a homomorphism $\Sigma : H_c^1(\mathbb{R}) \to \mathbb{Z}$. It is easy to check that Σ is a bijection, and therefore $H_c^1(\mathbb{R};\mathbb{Z}) \cong \mathbb{Z}$.

Note that $\{K \subseteq X \mid K \text{ compact}\}$ is a poset under inclusion. We obviously have that $K \subseteq L$ implies $X - K \supseteq X - L$, so for any inclusion $K \subseteq L$, we get a map $H^i(X, X-K) \to H^i(X, X-L)$, and this gives us directed system of abelian groups.

Theorem. *For any space X,*

$$H_c^i(X) \cong \varinjlim_K H^i(X, X-K).$$

Proof. Do on your own. ☐

It turns out that there is a relative fundamental class $[(M, M-K)] \in H_n(M, M-K)$, where M is an n-manifold. We can extend $\frown$ to $H_k(X, A) \times H^\ell(X, A) \to H_{k-\ell}(X, A)$, and then use the above theroem to define

$$\frown : H_k(X) \times H_c^\ell(X) \to H_{k-\ell}(X).$$

Proposition. *The cohomology with compact support of $\mathbb{R}^n$ is*

$$H_c^i(\mathbb{R}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if } i = n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $K_r = B_0(r)$. We get an increasing sequence of compact sets, $K_1 \subset K_2 \subset \cdots$, and hence a decreasing sequence $X - K_1 \supset X - K_2 \supset \cdots$. This is a cofinal sequence in the poset mentioned above. Because

$$H^i(\mathbb{R}^n, \mathbb{R}^n - B_0(r)) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

and all of the maps in the directed system are the identity, we will get the same thing when we take the algebraic limit. ☐

Lecture 24 (2012-11-26)

Today we'll be talking about the fundamental group. It may seem like it would be similar to homology, but in homology, we only deal with abelian / linear things, whereas the fundamental group is usually non-abelian.

The Fundamental Group

We start by considering maps from $[0,1]$ to X. If $\alpha, \beta : [0,1] \to X$ satisfy $\alpha(1) = \beta(0)$, then we can define their composition $(a * b) : [0,1] \to X$ by

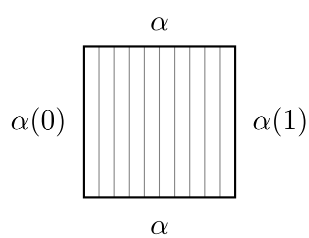
$$(a * b)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

We define a "path" to be a map from $[0,1]$ to X, with the endpoints treated specially. We define two paths to be homotopic when they are homotopic rel their endpoints.

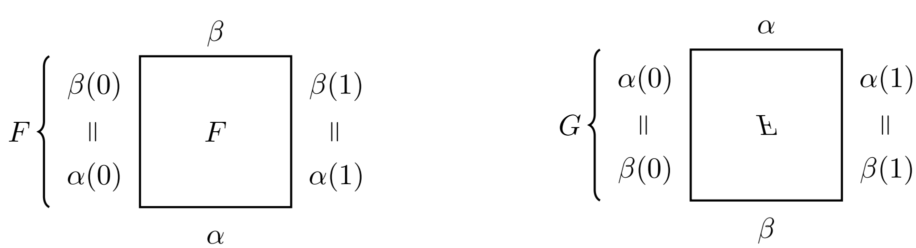
Thus, if α and β are paths with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, then we say $\alpha \sim \beta$ when there is some homotopy $F : [0,1]^2 \to X$ such that $F_0 = \alpha$, $F_1 = \beta$, $F_t(0) = \alpha(0) = \beta(0)$, and $F_t(1) = \alpha(1) = \beta(1)$.

Now we'll prove this is an equivalence relation.

For any path a, we have $a \sim a$ because we can define a homotopy as follows (imagine time progresses as one goes bottom to top; this homotopy is constant on the gray lines):



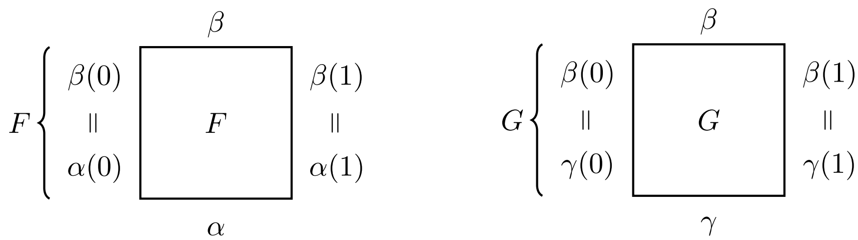
Suppose α and β are paths with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, such that $\alpha \sim \beta$ (say, via a path homotopy $F : [0,1]^2 \to X$ from $F_0 = \alpha$ to $F_1 = \beta$). Then we have $\beta \sim \alpha$, because we can define a path homotopy $G : [0,1]^2 \to X$ from $G_0 = \beta$ to $G_1 = \alpha$ simply as $G_t(s) = F_{1-t}(s)$. Pictorally, (again, imagine time progresses as one goes from bottom to top):

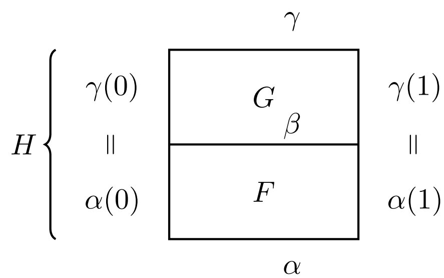


Lastly, if α, β, γ are paths with $\alpha(0) = \beta(0) = \gamma(0)$ and $\alpha(1) = \beta(1) = \gamma(1)$, such that $\alpha \sim \beta$ and $\beta \sim \gamma$ via homotopies $F : [0,1]^2 \to X$ from $F_0 = \alpha$ to $F_1 = \beta$ and $G : [0,1]^2 \to X$ from $G_0 = \beta$ to $G_1 = \gamma$, we have $\alpha \sim \gamma$ because we can define a homotopy $H : [0,1]^2 \to X$ from $H_0 = \alpha$ to $H_1 = \gamma$ by

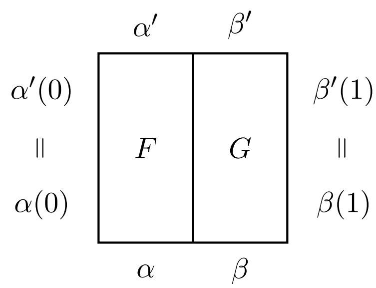
$$H_t(s) = \begin{cases} F_{2t}(s) & \text{if } 0 \le t \le \frac{1}{2}, \\ G_{2t-1}(s) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Pictorally,



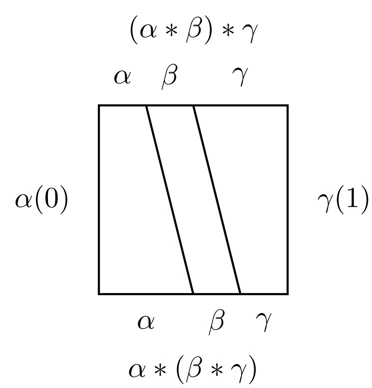


Note also that if $\alpha * \alpha'$ and $\beta \sim \beta'$, then we also have $(\alpha * \beta) \sim (\alpha' * \beta')$:

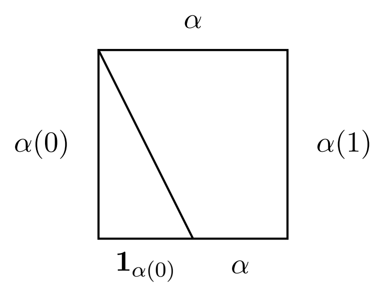


If $[\alpha]$ denotes the equivalence class of α with respect to $\sim$, then this demonstrates that $*$ descends to an operation on equivalence classes, i.e. $[\alpha * \beta] = [\alpha] * [\beta]$.

Moreover, $*$ is associative (when $*$ is defined) because there is a homotopy

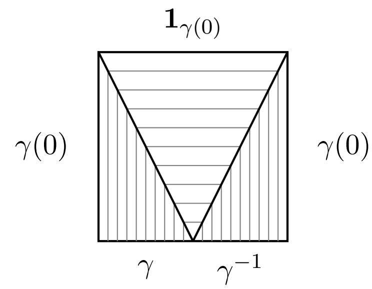


There are also "identities" for $*$ (be careful, because $*$ is only defined for paths whose endpoints are the same). We define $\mathbf{1}_p$ to be the constant map from $[0, 1]$ to $p \in X$. Then $\mathbf{1}_{\alpha(0)} * \alpha = \alpha$ because there is a homotopy



and similarly $\alpha * \mathbf{1}_{\alpha(1)} = \alpha$.

Lastly, there are inverses. If γ is a path, we define γ^{-1} to be the same path but with the opposite orientation, i.e. $\gamma^{-1}(t) = \gamma(1-t)$. Then $\gamma * \gamma^{-1} = \mathbf{1}_{\gamma(0)}$, because there is a homotopy



(it is constant on the gray lines), and similarly with $\gamma^{-1} * \gamma = \mathbf{1}_{\gamma(1)}$.

A structure satisfying these properties is called a groupoid. It is the same as a group, except there are some elements which may not be able to be composed.

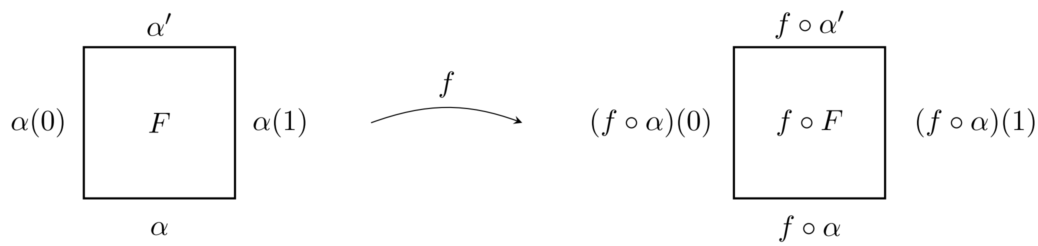
Thus, given any space X gives rise to a groupoid, called the fundamental groupoid of X. The construction of this groupoid didn't favor any point in X over any other, but if we break the symmetry a bit and choose some point $x \in X$ to be our basepoint, then we can define a group as follows: let $\pi_1(X, x)$ be the set of equivalence classes of paths $\alpha : [0,1] \to X$ which both start and end at x, i.e. $\alpha(0) = \alpha(1) = x$. Then $\pi_1(X, x)$ forms a group under the operation $*$, and the identity of $\pi_1(X, x)$ is just $\mathbf{1}_x$. (Note that we usually want to assume X is connected.) We say that $\pi_1(X, x)$ is the fundamental group of X with basepoint x.

How does $\pi_1(X, x)$ depend on our choice of x?

Given two points x and y, choosing a path γ from x to y determines a homomorphism from $\pi_1(X, x)$ to $\pi_1(X, y)$ by sending $[\alpha]$ to $[\gamma^{-1} * \alpha * \gamma]$. In fact, it is easy to see that it must be an isomorphism (take γ^{-1} as a path from y to x). If $x = y$, then this isomorphism is just the inner automorphism that is conjugation by $[\gamma]$, because $[\gamma^{-1} * \alpha * \gamma] = [\gamma^{-1}] * [\alpha] * [\gamma]$.

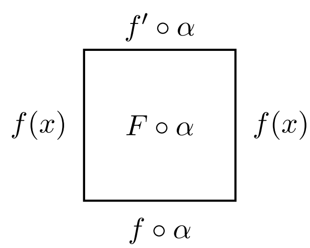
Thus, we often say somewhat loosely that a group is "the" fundamental group of a space X, even though the isomorphism is only determined up to inner automorphism. At any rate, the isomorphism class determined.

We also want π_1 to be functorial. Given a map $f : X \to Y$ and $\alpha : [0,1] \to X$, then $(f \circ \alpha) : [0,1] \to Y$ is a path in Y, so we get a map f_* sending paths in X to paths in Y, which is compatible with $\sim$ because f will also map any homotopy:



and moreover f_* is compatible with $*$. Thus, f_* induces a homomorphism of groupoids, and if we choose basepoints, we get a map $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$.

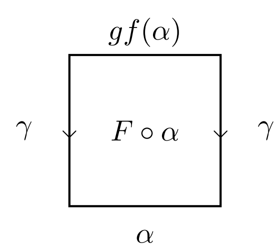
If f and f' are maps from X to Y both sending x to the same point, and $f \simeq f'$ rel x (i.e. there is a homotopy $F : X \times [0, 1] \to Y$ such that $F_t(x) = f(x) = f'(x)$), then the induced maps $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ and $f'_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ are in fact the same, because the homotopy F between f and f' gives us, for any loop α in X based at x, a homotopy



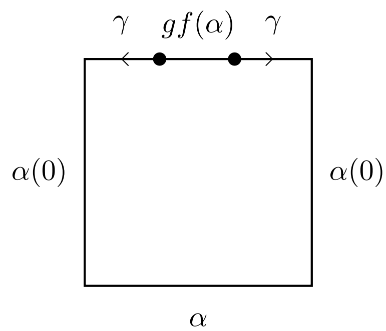
If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism. We can see this as follows. Let g be the homotopy inverse of f, so that there is a homotopy $F : X \times [0, 1] \to X$ such that $F_t(0) = (g \circ f)$ and $F_t(1) = \mathrm{id}_X$. Let γ keep track of what F does to x, so that $\gamma : [0, 1] \to X$ is a path starting at $(g \circ f)(x)$ and ending at x. Then the composition

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, gf(x)) \xrightarrow{\gamma_*} \pi_1(X, x)$$

is the identity map. Pictorally, we get immediately from our setup a homotopy (note that the left and right edges are **not** constant here, i.e. this is not a path-homotopy)



which we can then deform into a nice path-homotopy by shrinking the left and right edges onto the top, leaving the left and right sides to be mapped constantly to $\alpha(0)$:



In summary, for a connected space X with basepoint x, then $\pi_1(X, x)$ is a group, consisting of the homotopy calsses of paths based at x rel endpoints, with $*$ as the operation. Changing the basepoints gives isomorphic groups. Maps $f : X \to Y$ induce homomorphisms $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$. Homotopic maps induce the same homomorphism, and homotopy equivalences induce isomorphisms.

Lecture 25 (2012-11-28)

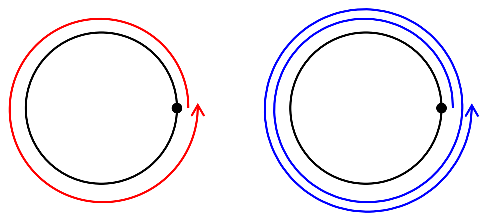
Recall that if X is a path-connected topological space and $x_0 \in X$, then $\pi_1(X, x_0)$ is the fundamental group of X with basepoint x_0. It consists of homotopy classes (rel endpoints) of loops $\alpha : [0,1] \to X$ satisfying $\alpha(0) = \alpha(1) = x_0$. The group operation is $*$.

Theorem. *If $f : X \to X$ is a homotopy equivalence, then $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x))$ is an isomorphism.*

Example. We know that $\mathbb{R}^2 \simeq \text{pt}$, so $\pi_1(\mathbb{R}^2, (0,0)) \cong \pi_1(\text{pt}, \text{pt}) = 1$.

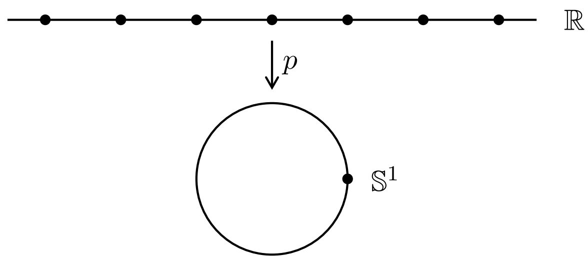
Today, we'll compute $\pi_1(\mathbb{S}^1, 1)$, and introduce covering spaces.

Are these two loops equivalent in $\pi_1(\mathbb{S}^1, 1)$?



No; they go around different numbers of times. It's clear intuitively that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. But how can we prove this?

The right way to think about $\mathbb{S}^1$ is as $\mathbb{R}/\mathbb{Z}$, under the quotient $p : \mathbb{R} \to \mathbb{S}^1$ defined by $p(t) = e^{2\pi i t}$.

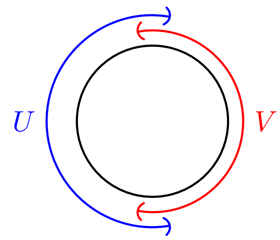


Note that we can "lift" paths in $\mathbb{S}^1$ up to $\mathbb{R}$, as long as we have specified where to start.

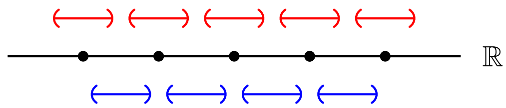
Theorem. *For any $\alpha : [0,1] \to \mathbb{S}^1$ with $\alpha(0) = x_0$, and any choice of $\widetilde{x_0} \in p^{-1}(x_0)$, then there is a unique $\widetilde{\alpha} : [0,1] \to \mathbb{R}$ with $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = \widetilde{x_0}$.*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {} \arrow[ur, "\widetilde{\alpha}"] \arrow[r, "\alpha"'] & \mathbb{S}^1 \end{tikzcd}$$

Proof. We can cover $\mathbb{S}^1$ by open U, V like this:



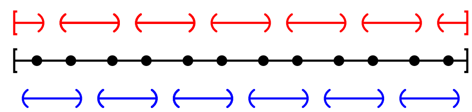
Let $\{U_i\}, \{V_i\}$ be the connected components of $p^{-1}(U)$ and $p^{-1}(V)$, respectively.



Note that, for each U_i and V_i, the maps $p|_{U_i} : U_i \to U$ and $p|_{V_i} : V_i \to V$ are all homeomorphisms, so at least *locally*, there are lifts; indeed for any space Z and any map $f : Z \to U$, there is a unique lift $\widetilde{f}$, i.e. a map such that $p \circ \widetilde{f} = f$.

$$\begin{tikzcd} & U_i \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & U \end{tikzcd}$$

Now pull back U and V under α to cover $[0, 1]$ (which has a finite subcover because it is compact).

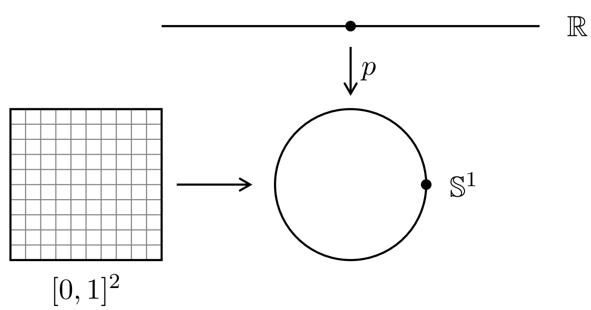


Thus, α breaks into subpaths α_i, with $\alpha_i : [v_{i-1}, v_i] \to U$ (or V). Therefore, there is a unique lift $\widetilde{\alpha_1} : [0, v_1] \to U_i \subseteq \mathbb{R}$, where this is the unique U_i containing $\widetilde{x_0}$. Then there is a unique lift $\widetilde{\alpha_2} : [v_1, v_2] \to V_j \subseteq \mathbb{R}$ such that $\widetilde{\alpha_2}(v_1) = \widetilde{\alpha_1}(v_1)$. We then proceed inductively. $\square$

Thus, we've shown that we can lift paths. We can also lift homotopies:

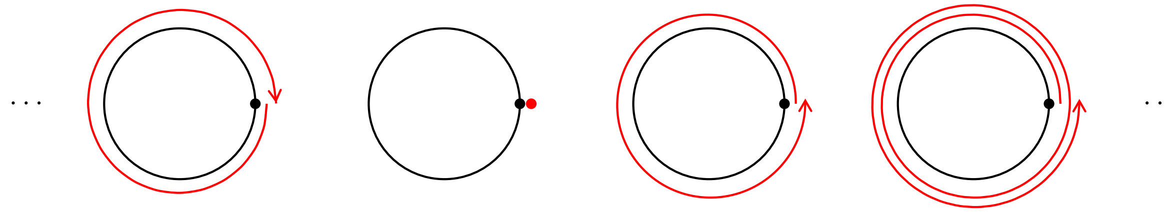
Theorem. *For any $H_s(t) : [0, 1]^2 \to \mathbb{S}^1$ such that $H_0(0) = x_0$, and any choice of $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{H}_s(t) : [0, 1]^2 \to \mathbb{R}$ such that $\widetilde{H}_0(0) = \widetilde{x_0}$ and $p \circ \widetilde{H} = H$.*

Proof. Break $[0, 1]^2$ into squares small enough to guarantee that each maps into either U or V, and lift each square inductively in the same way as our earlier proof.



$\square$

Now let's get back to $\pi_1(\mathbb{S}^1, 1)$. Define $\gamma_n : [0,1] \to \mathbb{S}^1$ by $\gamma_n(t) = e^{2\pi int}$.



Note that the unique lift $\widetilde{\gamma_n}$ with $\widetilde{\gamma_n}(0) = 0$ is

Define $\Phi : \mathbb{Z} \to \pi_1(\mathbb{S}^1, 1)$ by $\Phi(n) = [\gamma_n]$.

Theorem. *The map Φ is an isomorphism.*

Proof. First, let's show that Φ is surjective. Given $[\alpha] \in \pi_1(\mathbb{S}^1, 1)$, by our earlier theorem, there is a unique lift $\alpha : [0,1] \to \mathbb{R}$ with $\widetilde{\alpha}(0) = 0$. Then $\widetilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$, so $\widetilde{\alpha}(1) = n$ for some $n \in \mathbb{Z}$. Obviously, we are going to want to define a homotopy between α and γ_n.

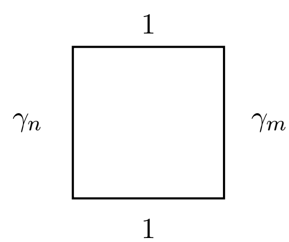
Define $H : [0,1]^2 \to \mathbb{R}$ by $H_s(t) = snt + (1-s)\widetilde{\alpha}(t)$. Note that

$$H_0(t) = \widetilde{\alpha}(t), \quad H_1(t) = nt = \widetilde{\gamma_n}(t), \quad H_s(0) = 0, \quad H_s(1) = n.$$

Thus, $p \circ H$ is a homotopy $\alpha \sim \gamma_n$, so $[\alpha] = [\gamma_n]$.

Now we'll show that Φ is injective. Suppose that $[\gamma_m] = [\gamma_n]$, so that there is some $H : [0,1]^2 \to \mathbb{S}^1$ such that

$$H_0(t) = \gamma_n(t), \quad H_1(t) = \gamma_m(t), \quad H_s(0) = 1, \quad H_s(1) = 1.$$



We lift this H to $\widetilde{H} : [0,1]^2 \to \mathbb{R}$, with $\widetilde{H}_0(0) = 0$. Then we have $\widetilde{H}_0(t) = \widetilde{\gamma_n}(t)$ and $\widetilde{H}_1(t) = \widetilde{\gamma_m}(t)$, and note that $\widetilde{H}_s(0)$ and $\widetilde{H}_s(1)$ have to always map into $p^{-1}(1) = \mathbb{Z}$, and therefore they must be constant. Thus,

$$m = \widetilde{\gamma_m}(1) = \widetilde{H}_1(1) = \widetilde{H}_0(1) = \widetilde{\gamma_n}(1) = n. \qquad \square$$

Definition. If $\widetilde{X}$ and X are topological spaces, we say that the map $p : \widetilde{X} \to X$ is a covering map when there is an open cover $\{V_i\}$ of X such that $p|_U$ is a homeomorphism for any connected component U of $p^{-1}(V_i)$, for any V_i. We say that $\widetilde{X}$ is a covering space of X.

Lecture 26 (2012-11-30)

Recall that last time, we proved that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, and introduced covering spaces.

Free products of groups

Definition. Let G and H be groups. We define $G * H$ to be the group generated by G and H, with no extra relations; thus,

$$G * H = \{g_1 h_1 \cdots g_k h_k \mid g_i \in G, h_i \in H\},$$

and concatenation of words is the group operation. It is a theorem that $G * H$ is in fact a group; associativity is the only tricky part. We can also define, for any collection of groups G_α, their free product $\mathop{\ast}_\alpha G_\alpha$.

Example. The free group on $S = \{s_1, \dots, s_n\}$ is just $\mathbb{Z} * \cdots * \mathbb{Z}$, with one copy of $\mathbb{Z}$ for each s_i.

Theorem (Universal property of free products)**.** *For any collection $\{G_\alpha\}$ of groups and homomorphisms $\{\phi_\alpha : G_\alpha \to H\}$, there is a unique homomorphism $\phi : \mathop{\ast}_\alpha G_\alpha \to H$ such that $\phi(g) = \phi_\alpha(g)$ for any $g \in G_\alpha$ for all α.*

Recall that we specify a presentation of a group as

$$G = \langle \underbrace{g_1, \dots, g_k}_{\text{relations}} \mid \underbrace{r_1, \dots, r_m}_{\substack{\text{relations, i.e.} \\ \text{words in the } g_i}} \rangle.$$

What this really means is

$$G = F(\{g_1, \dots, g_k\}) / \langle\!\langle r_1, \dots, r_m \rangle\!\rangle,$$

where $\langle\!\langle \cdot \rangle\!\rangle$ denotes the normal subgroup generated by a set. Note that we need to use this normal closure because if b is a relation we want to kill, we also need to kill aba^{-1}.

Free products with amalgamation

Suppose we have (not necessarily injective) group homomorphisms as follows:

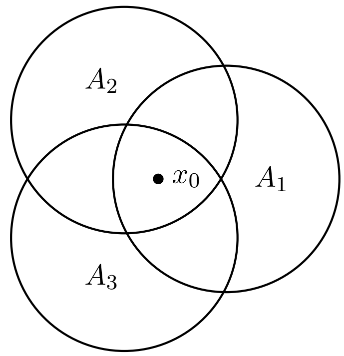
$$\begin{tikzcd} & A_1 \\ H \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & A_2 \end{tikzcd}$$

Then $A_1 *_H A_2$, the free product of A_1 and A_2 along H, is defined to be

$$A_1 *_H A_2 = A_1 * A_2 / \langle\!\langle \{i_1(h) i_2(h)^{-1} \mid h \in H\} \rangle\!\rangle.$$

The van Kampen theorem

Suppose that $X = \bigcup_\alpha A_\alpha$, where each of the A_α are path connected and open, and where there is some $x_0 \in \bigcap_\alpha A_\alpha$. For example, we might have



For any α and β, we have a commutative diagram of topological spaces

$$\begin{tikzcd} & A_\alpha \arrow[dr, "j_\alpha"] & \\ A_\alpha \cap A_\beta \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & X \\ & A_\beta \arrow[ur, "j_\beta"'] & \end{tikzcd}$$

which induces a commutative diagram of fundamental groups (all with respect to the basepoint x_0):

$$\begin{tikzcd} & \pi_1(A_\alpha) \arrow[dr, "j_\alpha"] & \\ \pi_1(A_\alpha \cap A_\beta) \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & \pi_1(X) \\ & \pi_1(A_\beta) \arrow[ur, "j_\beta"'] & \end{tikzcd}$$

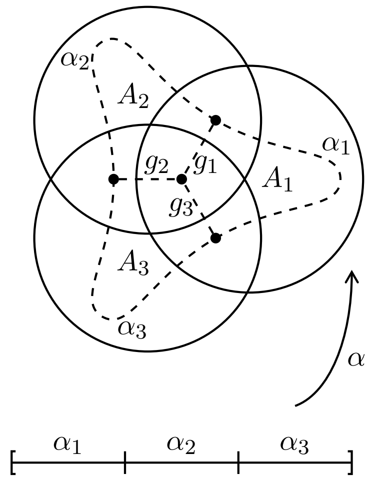
Clearly, we have that $j_{\alpha*}(i_{\alpha\beta_*}([\gamma])) = j_{\beta_*}(i_{\beta\alpha_*}([\gamma])$. There is a map $\Phi : \ast_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ induced by taking the $j_{\alpha*}$ together.

Theorem (van Kampen)**.** *With this notation, if $A_\alpha \cap A_\beta$ is path connected for any α, β, then Φ is surjective. If $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected for any α, β, γ, then* $\ker(\Phi)$ *is generated by $i_{\alpha\beta_*}([\gamma]) i_{\beta\alpha_*}([\gamma])^{-1}$ for any loop $\gamma \in A_\alpha \cap A_\beta$.*

Theorem (Simpler version)**.** *Suppose that $X = A_1 \cup A_2$ such that $A_1 \cap A_2$ is path-connected. Then*

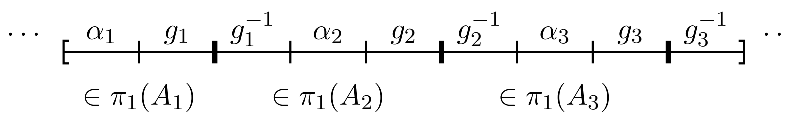
$$\pi_1(X) \cong \pi_1(A_1) *_{\pi_1(A_1 \cap A_2)} \pi_1(A_2).$$

Sketch of proof.



First, let's prove that Φ is surjective. Given $[\alpha] \in \pi_1(X)$, pull back the open cover A_α via α (terrible notation!) and split the loop α into subpaths α_i, each of which maps into some A_α.

Now rewrite this as



Now, let's try to understand the kernel $\ker(\Phi)$. Given two elements of $\ast_\alpha \pi_1(A_\alpha)$ which are sent to the same thing in $\pi_1(X)$, say

$$[f_1]\cdots[f_k] = [f_1']\cdots[f_m'] \in \pi_1(X).$$

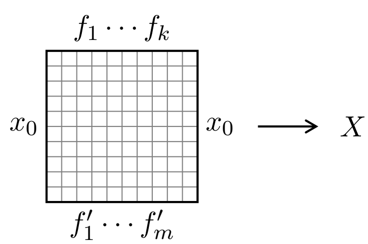
Then we have

$$[f_1\cdots f_k] = [f_1'\cdots f_m'] \in \pi_1(X),$$

and we want to show that we can get from $[f_1\cdots f_k]$ to $[f_1'\cdots f_m']$ by

1. combining $[f_i][f_{i+1}] = [f_i f_{i+1}]$ if $f_i, f_{i+1} \in A_\alpha$, or
2. taking $[f_i]$ to be in either $\pi_1(A_\alpha)$ or $\pi_1(A_\beta)$ if $[f_i] \in \pi_1(A_\alpha \cap A_\beta)$.

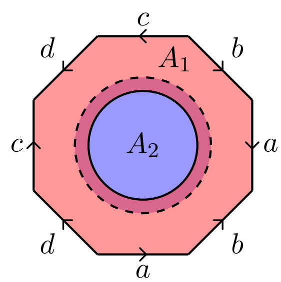
We can create a homotopy from $f_1\cdots f_k$ to $f_1'\cdots f_m'$ by dividing up $[0,1]^2$ into a grid such that, as we pass over each line in the grid, we are using either operation 1 or operation 2.

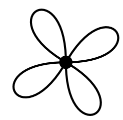


$\square$

Example.

- Let's compute $\pi_1(\Sigma_2)$, where Σ_2 is the two-holed torus. Recall that we can write Σ_2 as the quotient of an octagon. We break up the octagon into two pieces, A_1 and A_2:



$A_1 \simeq$ 

We have

$$\pi_1(\Sigma_2) = (\pi_1(A_1) * 1)/\pi_1(A_1 \cap A_2) = \langle a, b, c, d \rangle / \mathbb{Z},$$

but we need to understand which $\mathbb{Z}$. Looking at $A_1 \cap A_2$, we see that

$$\pi_1(\Sigma_2) = \langle a, b, c, d \rangle / [a, b][c, d].$$

Lecture 27 (2012-12-03)

Theorem (Gordon-Luecke, 1980's)**.** *Let K_1, K_2 be knots in $\mathbb{R}^3$. Then K_1 is equivalent to K_2, i.e. there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(K_1) = K_2$, if and only if $\pi_1(\mathbb{R}^3 \setminus K_1) \cong \pi_1(\mathbb{R}^3 \setminus K_2)$.*

(warning: this is an incorrect statement of the theorem)

Comparing H_1 and π_1

Let Γ be any group. The commutator subgroup of Γ is defined to be

$$[\Gamma, \Gamma] = \langle \{ghg^{-1}h^{-1} \mid g, h \in \Gamma\}\rangle.$$

Then $\Gamma/[\Gamma, \Gamma]$ is abelian, and $\Gamma/[\Gamma, \Gamma] \cong \Gamma^{\text{ab}}$, where Γ^{ab} is by definition the unique group satisfying the following universal mapping property: for any abelian group A and homomorphism $\phi : \Gamma \to A$, there is a unique $\overline{\phi} : \Gamma^{\text{ab}} \to A$ such that

$$\begin{tikzcd} \Gamma \arrow[r, "\phi"] \arrow[d, "\pi"'] & A \\ \Gamma^{\text{ab}} \arrow[ur, dashed, "\overline{\phi}"'] & \end{tikzcd}$$

You proved the following theorem on the most recent homework:

Theorem. *Let X be any path-connected space. Then the map $\phi : \pi_1(X) \to H_1(X, \mathbb{Z})$ defined by $\phi([\gamma]) = \gamma_*([\mathbb{S}^1])$ induces an isomorphism*

$$\overline{\phi} : \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X, \mathbb{Z}).$$

Covering Spaces

Throughout, let X and Y be path-connected and locally path-connected spaces.

Definition. A map $p : Y \to X$ is a covering space when any $x \in X$ has a neighborhood U_x such that, for each $x_\alpha \in p^{-1}(x)$, there is a neighborhood V_α of x_α such that $p|_{V_\alpha} : V_\alpha \to U_x$ is a homeomorphism, and $V_\alpha \cap V_\beta$ for $\alpha \neq \beta$.

Example. A group action on X is just a subgroup $\Gamma \subset \text{Homeo}(X)$. The action is discrete if, for all $x \in X$, there is a neighborhood U_x of x such that for all $g \in \Gamma \setminus \{\text{id}\}$, $gU_x \cap U_x = \varnothing$. You can check that if Γ acts discretely on X, then the quotient $X \to X/\Gamma$ is a covering map.

To give explicit examples of this, we can take $\Gamma \subset \text{Homeo}(\mathbb{R}^2)$ to be

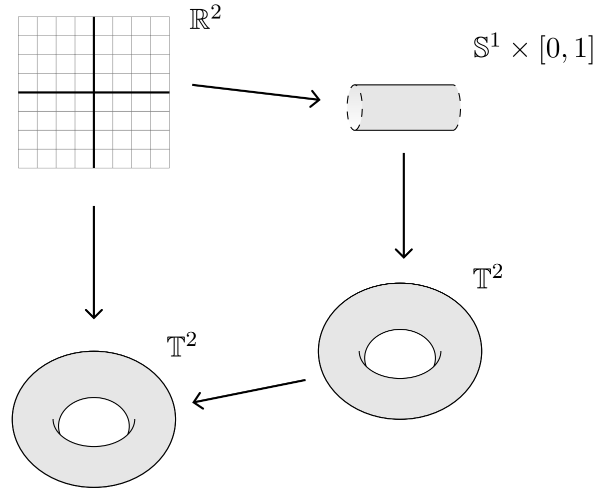
$$\Gamma = \langle (x, y) \mapsto (x+1, y),\ (x, y) \mapsto (x, y+1)\rangle,$$

in which case we get a covering map $\mathbb{R}^2 \to \mathbb{R}^2/\Gamma \cong \mathbb{T}^2$. We could also consider

$$\Lambda = \langle (x, y) \mapsto (x, y+1)\rangle,$$

in which case we get a covering map $\mathbb{R}^2 \to \mathbb{R}^2/\Gamma \cong \mathbb{S}^1 \times \mathbb{R}$. Quotienting the rest of the way produces a covering map $\mathbb{S}^1 \times \mathbb{R} \to \mathbb{T}^2$. In fact, $\mathbb{T}^2$ covers itself; for any $a, b \in \mathbb{Z}^2$, the map which acts as a degree a map on the first factor of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and as a degree b map on the second factor produces a covering map $\mathbb{T}^2 \to \mathbb{T}^2$.

Note that there are lots of different covering maps of $\mathbb{T}^2$, and some of them cover others:



We can build a dictionary between the group theory of $\pi_1(X)$ and the covering space theory of X. It is a Galois correspondence. Technically, we need to assume that X is path-connected, locally path-connected, and semi-locally simply connected.

$\pi_1(X)$	$\longleftrightarrow$	X
subgroups $H \subseteq \pi_1(X)$	$\longleftrightarrow$	covering spaces $p : Y \to X$
$\pi_1(X)$	$\longleftrightarrow$	$\mathrm{id} : X \to X$
$\{e\}$	$\longleftrightarrow$	universal cover $\widetilde{X} \to X$
$H \subset \pi_1(X)$	$\longleftrightarrow$	$\widetilde{X}/H \to X$
normal subgroups	$\longleftrightarrow$	$p : Y \to X$ is regular
$[\pi_1(X) : H]$	$\longleftrightarrow$	$\#p^{-1}(x)$, i.e. number of sheets
conjugacy classes of elements	$\longleftrightarrow$	free homotopy classes of loops

Theorem. *If Γ acts discretely on Y and $\pi_1(Y) = 0$, then $\pi_1(Y/\Gamma) \cong \Gamma$.*

Example. We can cover the unit disk model of $\mathbb{H}^2$ with octagons, and act discretely on it by $\Gamma = \langle x, y, z, w\rangle$ for some transformations x, y, z, w which identify the appropriate sides of the octagons, and we get $\mathbb{H}^2/\Gamma \cong \Sigma_2$.

Example. If $L(p, q)$ is the lens space for p and q, we can obtain it as $\mathbb{S}^3/(\mathbb{Z}/p\mathbb{Z})$.

Non-example. Let $r \in \mathbb{R} \setminus \mathbb{Q}$, and let $\mathbb{Z}$ act on $\mathbb{S}^1$ by $e^{2\pi i\theta} \mapsto e^{2\pi i(\theta+r)}$. This action is not discrete, and $\mathbb{S}^1/\mathbb{Z}$ is not even Hausdorff.

Non-example. Let $\mathbb{Z}/3\mathbb{Z}$ act on $\mathbb{D}^2$ by rotations by $2\pi/3$. This action is not discrete, though it almost is - it would be discrete if we threw away the origin. This kind of situation is called a branched cover.

Proposition. *Let $p : Y \to X$ be a covering map. Then $p_* : \pi_1(Y, y) \to \pi_1(X, p(y))$ is injective.*

Proof. The idea is clear: if a loop $p_*([\gamma])$ is trivial in $\pi_1(X, p(y))$, we can lift the homotopy to one demonstrating that $[\gamma]$ is trivial. $\square$

Lecture 28 (2012-12-05)

Applications to Knots

Proposition. *The trefoil knot is knotted.*

Proof. We know that $\pi_1(\mathbb{S}^3 - \text{unknot}) \cong \mathbb{Z}$, but it is not too hard to show that $\Gamma = \pi_1(\mathbb{S}^3 - \text{trefoil}) = \langle a, b \mid aba = bab \rangle$. We claim that Γ is not isomorphic to $\mathbb{Z}$. To see this, note that there is a surjection $f : \Gamma \to S_3$ (a non-abelian group), defined by $a \mapsto (12)$ and $b \mapsto (23)$, because these elements of S_3 satisfy the relation $aba = bab$. $\square$

Let $T(p, q)$ be the (p, q)-torus knot, i.e. the knot obtained by embedding the torus $\mathbb{T}^2$ in $\mathbb{R}^3$ in the standard way and taking the curve on the torus that wraps around p times one way and q times the other way.

Proposition. *$T(p, q)$ is equivalent to $T(m, n)$ $\iff$ $m = p$ and $n = q$, or $m = q$ and $n = p$.*

Proof. It turns out that $\Gamma = \pi_1(\mathbb{S}^3 - T(p, q)) = \langle a, b \mid a^p b^q \rangle$. Thus, a^p commutes with both a and b, so that $a^p \in Z(\Gamma)$, hence $\langle a^p \rangle \triangleleft \Gamma$. It is easy to see that $\Gamma/\langle a^p \rangle \cong \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, hence $(\Gamma/\langle a^p \rangle)^{\text{ab}} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, which allows you to distinguish a lot of things. $\square$

Proposition. *If $p : Y \to X$ is a cover with $p(y) = x$, then $p_* : \pi_1(Y, y) \to \pi_1(X, x)$ is an injection, and $p_*(\pi_1(Y, y))$ consists of those $[\gamma]$ such that any lift of γ to Y is a loop.*

Proof. The map sends γ to $\alpha = p \circ \gamma$, and if α is trivial in $\pi_1(X, x)$ there is some homotopy α_t from α to a constant map. Lifting this homotopy to Y gives a homotopy from γ to a constant map. $\square$

Proposition. *Let $p : Y \to X$ be a covering map, with X, Y connected. Then $\#p^{-1}(X) = [\pi^{-1}(X) : p_*(\pi_1(Y))]$.*

Proof. Define $\varphi : \pi_1(X)/p_*(\pi_1(Y)) \to \{p^{-1}(x)\}$ by sending the coset of $[\gamma]$ to $\gamma(1)$. Show that this map is bijective. $\square$

Note that $\pi_1(X)$ acts on the set $p^{-1}(x)$ via $[\gamma] \cdot y \mapsto \widetilde{\gamma}(1)$, where $\widetilde{\gamma}$ is the lift of γ starting at y. For example, $\pi_1(\mathbb{S}^1)$ acts on $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$ by translations.

The stabilizer $\text{Stab}(y)$ is conjugate to $p_*(\pi_1(Y))$.

Definition. We say that two coverings $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are isomorphic when there is a homeomorphism $f : Y_1 \to Y_2$ such that $p_2 = p_1 \circ f$, i.e. f takes the set $p_1^{-1}(x)$ to $p_2^{-1}(x)$ for any $x \in X$. For any covering $p : Y \to X$, the group of deck transformations of p is just $\text{Aut}(p : Y \to X)$.

Example. The group of deck transformations of the covering $\mathbb{R} \to \mathbb{S}^1$ is just $\mathbb{Z}$, consisting of the translations of $\mathbb{R}$ by integers.

As we mentioned before, there is a correspondence

$$\{\text{subgroups } H \subseteq \pi_1(X, x)\} \longleftrightarrow \{\text{coverings } p : Y \to X \text{ up to isomorphism}\}.$$

In particular, the trivial subgroup $\{e\}$ corresponds to a simply-connected cover $\widetilde{X}$ of X, called the universal cover of X. You can look in Hatcher for details, but essentially $\widetilde{X}$ consists of certain equivalence classes of paths in X.

Looking at universal covers for graphs is good practice for the general case.

Definition. A covering map $p : Y \to X$ is called a regular cover (a.k.a. normal cover) when for all $x \in X$ and $y_1, y_2 \in p^{-1}(x)$, there is a deck transformation $h \in \mathrm{Aut}(p : Y \to X)$ such that $h(y_1) = y_2$.

It turns out that $p : Y \to X$ is regular if and only if $p_*(\pi_1(Y)) \triangleleft \pi_1(X)$, i.e. for any $h \in p_*(\pi_1(Y))$ and $g \in \pi_1(X)$, we have $ghg^{-1} \in p_*(\pi_1(Y))$. We can see this because $ghg^{-1}(g(y)) = gh(y) = g(y)$, since $h \in \pi_1(Y)$, so that $ghg^{-1} \in \mathrm{Stab}(g(y)) \cong \pi_1(Y, g(y))$. This is a special case of a very important general fact:

$$\mathrm{Fix}(ghg^{-1}) = g\mathrm{Fix}(h).$$

The exam will cover only the things we've done in class, but you should be sure to learn the lifting criterion on your own.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[r, "f"'] \arrow[ur, dashed, "?"] & X \end{tikzcd} \text{ can lift } \iff f_*(\pi_1(Z)) \subseteq p_*(\pi_1(Y))$$